A duality theorem for Tate-Shafarevich groups of curves over algebraically closed fields

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Abstract

In this note, we prove a duality theorem for the Tate-Shafarevich group of a finite discrete Galois module over the function field K of a curve over an algebraically closed field: There is a perfect duality of finite groups $Ш^1(K,F) \times Ш^1(K,F') \to \mathbf{Q}/\mathbf{Z}$ for F a finite étale Galois module on K of order invertible in K and with $F' = \mathrm{Hom}(F, \mathbf{Q}/\mathbf{Z}(1))$.

Furthermore, we prove that $\mathrm{H}^1(K,G) = 0$ for G a simply connected, quasisplit semisimple group over K not of type E_8.

Keywords: Galois cohomology; Étale and other Grothendieck topologies and cohomologies

MSC 2010: 11S25, 14F20

1 Introduction

The result is proved along the lines of [HS16]. The statement basically follows from Poincaré duality for non-empty open subschemes of smooth proper curves Theorem 2.2 and a local duality result Proposition 3.1 for the completions of the quotient fields of the local rings at closed points for such a curve.

Notation. If A is an Abelian group, denote by $A^{(p')}$ the prime-to-p part of A. Denote the cokernel of $A \xrightarrow{n} A$ by A/n and its kernel by $A[n]$, and the p-primary subgroup $\varinjlim_n A[p^n]$ by $A[p^\infty]$. Henselisation of a ring A is denoted by A^h. The maximal unramified extension of a discretely valued field K is denoted by K^{nr}. Canonical isomorphisms are often denoted by "=". If not stated otherwise, all cohomology groups are taken with respect to the étale topology. We denote Pontryagin duality by $(-)^D$ and duals of étale sheaves by $(-)^\vee$. By $X^{(i)}$, we denote the set of codimension-i points of a scheme X, and by $|X|$ the set of closed points. The residue field of a point $v \in X$ is denoted by $\kappa(v)$.

2 Preliminaries

Let k be an algebraically closed field of characteristic p and C/k a smooth projective geometrically integral curve. Let $K = k(C)$ be the function field of C. One has $\mathrm{cd}(K) = 1$ by [NSW08], p. 367, Theorem 6.5.14 and p. 365, Proposition 6.5.10. For $v \in C^{(1)} = |C|$, let K_v be the completion of K with respect to the discrete valuation v with residue field $\kappa(v) = k$. Note that K_v is a complete discrete valued field with algebraically closed residue field, so $\mathrm{dd}(K_v) = 1$ by a theorem of Lang [NSW08], p. 364, Theorem 6.5.6, so $\mathrm{Br}(K_v) = 0$ by [NSW08], p. 365, Proposition 6.5.8 and $\mathrm{cd}(K_v) = 1$ by [NSW08], p. 366, Proposition 6.5.11. One has $K_v \cong k((t))$ by the existence of a coefficient field for equal characteristic complete discrete valuation rings [Ser79], p. 33, Theorem II.4.2.

Theorem 2.1 (Poincaré duality)**.** *Let $U \subseteq C$ be a non-empty open subscheme. Then for any constructible sheaf $\mathscr{F}$ of order invertible on U, one has a perfect pairing of finite groups*

$$\mathrm{H}^r_c(U,\mathscr{F}) \times \mathrm{Ext}^{2-r}_U(\mathscr{F}, \mathbf{Q}/\mathbf{Z}(1)) \to \mathrm{H}^2_c(U, \mathbf{Q}/\mathbf{Z}(1)^{(p')}) = (\mathbf{Q}/\mathbf{Z})^{(p')}.$$

Proof. Poincare duality [Mil80], p. 175, Theorem V.2.1. □

Theorem 2.2 (Poincaré duality)**.** *Let $U \subseteq C$ be a non-empty open subscheme. Then for any locally constant constructible sheaf $\mathscr{F}$ of order invertible on U, there is an isomorphism $\mathrm{Ext}_U^{2-r}(\mathscr{F}, \mathbf{Q}/\mathbf{Z}(1)) = \mathrm{H}^{2-r}(U, \mathscr{F}^\vee(1))$ of finite groups and a perfect pairing of finite groups*

$$\mathrm{H}_c^r(U, \mathscr{F}) \times \mathrm{H}^{2-r}(U, \mathscr{F}^\vee(1)) \to \mathrm{H}_c^2(U, \mathbf{Q}/\mathbf{Z}(1)) = (\mathbf{Q}/\mathbf{Z})^{(p')}.$$

Proof. See [Fu11], p. 432, Corollary 8.3.2. □

Note that a sheaf represented by a finite étale group scheme is locally constant constructible by [Fu11], p. 246, Proposition 5.8.1 (i).

Proposition 2.3. *Let X be an integral locally Noetherian scheme and G_K be a finite étale group scheme over the function field K of X. Then there is a non-empty open subscheme U of X such that G_K extends to a finite étale group scheme G on U.*

Proof. There is a finite Galois extension L/K such that $G_K \times_K L$ becomes a constant group scheme. Let $f : \tilde{X} \to X$ be the normalisation of X in L. By [GW10], p. 342, Corollary 12.52 (1), a normalisation morphism of varieties is finite. Since X is locally Noetherian, by [Liu06], p. 154, Corollary 4.4.12, there is a non-empty open subscheme U of X such that $f^{-1}(U) \to U$ is a finite étale Galois covering with Galois group Γ. Since f is fpqc, there is an equivalence of categories between quasi-coherent $\mathscr{O}_U$-modules and quasi-coherent $\mathscr{O}_{f^{-1}(U)}$-modules with Γ-action, see [BLR90], p. 139, Example B.

We are looking for a $U'' \subseteq U$ open such that the Γ-action on L^r, r the rank of G_K, extends to $\mathscr{O}_{f^{-1}(U)}^r$, compatible with the group scheme structure. Without loss of generality, one can assume $X = \mathrm{Spec}(A)$ and $\tilde{X} = \mathrm{Spec}(B)$ affine. A group element $\gamma \in \Gamma$ acting on L^r corresponds to a matrix $\gamma \in \mathrm{Mat}_{r\times r}(L) = \mathrm{Mat}_{r\times r}(B \otimes_A K)$. Hence, there is an open affine $D(s_\gamma) \subseteq A$ for an $s_\gamma \in A$ such that $\gamma \in \mathrm{Mat}_{r\times r}(B \otimes_A A_{s_\gamma})$. Let U' be the intersection of the $D(s_\gamma)$ for all $\gamma \in \Gamma$ finite. There is a $U'' \subseteq U'$ open such that the conditions for a group action are satisfied and the group scheme structure is preserved. By descent theory, the claim follows. □

3 The Tate-Shafarevich group

Proposition 3.1 (local duality)**.** *Let F be a finite Galois module of order n invertible in K_v. Then there is a perfect duality of finite groups*

$$\mathrm{H}^0(K_v, F) \times \mathrm{H}^1(K_v, \mathrm{Hom}(F, \mu_n)) \to \mathbf{Z}/n.$$

Proof. This follows from [NSW08], p. 186, Duality Theorem 3.4.6 since $\mathrm{cd}(K_v) = 1$ and

$$D_0(\mathbf{Z}/p) = \varinjlim_{U} \mathrm{H}^0(U, \mathbf{Z}/p)^D = \varinjlim_{U} \mathbf{Z}/p = 0,$$

the direct limit taken over the corestriction maps which are given by multiplication by $[G_K : U]$ on H^0. Note that the dualising module $D = \mathbf{Q}/\mathbf{Z}^{(p')} \cong \mathbf{Q}/\mathbf{Z}^{(p')}(1)$ since $\mu_n \subset K^\times$. See also [NSW08], p. 188, Exercise 3. □

Definition 3.2. *Let F be a discrete Galois module. The* **Tate-Shafarevich group** *$Ш^r(K, F)$ is defined as*

$$Ш^r(K, F) = \ker \Big(\mathrm{H}^r(K, F) \to \prod_{v \in |C|} \mathrm{H}^r(K_v, F)\Big).$$

Definition 3.3. *Let F be a discrete K-module with extension $\mathscr{F}$ to a non-empty open subscheme U of C. Define*

$$\mathcal{D}^r(U, \mathscr{F}) := \mathrm{im}\big(\mathrm{H}_c^r(U, \mathscr{F}) \to \mathrm{H}^r(U, \mathscr{F}) \to \mathrm{H}^r(K, F)\big).$$

Note that the $\mathcal{D}^r(U, \mathscr{F})$ are finite by Theorem 2.1, and that $\mathcal{D}^r(U', \mathscr{F}) \subseteq \mathcal{D}^r(U, \mathscr{F})$ for $U' \subseteq U$ open by covariant functoriality of $\mathrm{H}_c^r(-, \mathscr{F})$.

Proposition 3.4. *Let U be an open subscheme of C and $\mathscr{F}$ an étale sheaf on U. Then for $i \geq 1$, there is a long exact cohomology sequence*

$$\ldots \to \mathrm{H}_c^i(U, \mathscr{F}) \to \mathrm{H}^i(U, \mathscr{F}) \to \bigoplus_{v \in C\setminus U} \mathrm{H}^i(K_v, \mathscr{F}_v) \to \mathrm{H}_c^{i+1}(U, \mathscr{F}) \to \ldots$$

with $\mathscr{F}_v$ the pullback of $\mathscr{F}$ by the natural map $\operatorname{Spec} K_v \to U$*. If $\mathscr{F}_v$ is finite, the sequence is also exact for $i = 0$.*
Furthermore, for $V \subseteq U$ open, there is a long exact sequence

$$\ldots \to \mathrm{H}^i_c(V, \mathscr{F}) \to \mathrm{H}^i_c(U, \mathscr{F}) \to \bigoplus_{v \in U \setminus V} \mathrm{H}^i(\kappa(v), i_v^* \mathscr{F}) \to \mathrm{H}^{i+1}_c(V, \mathscr{F}) \to \ldots$$

with $i_v : \operatorname{Spec} \kappa(v) \hookrightarrow U$ *the closed immersion.*

Proof. Note that the natural maps $\mathrm{H}^i(K_v^h, \mathscr{F}_v) \to \mathrm{H}^i(K_v, \mathscr{F}_v)$ are isomorphisms for $i \geq 1$ by [HS05], p. 102, Lemma 2.7. If $\mathscr{F}_v$ is finite, Greenberg's approximation theorem cited in *loc. cit.* shows that $\mathrm{H}^0(K_v^h, \mathscr{F}_v) \to \mathrm{H}^0(K_v, \mathscr{F}_v)$ is an isomorphism. Then the claim is proved as in [Mil86], p. 167, Lemma II.2.4.
The second exact sequence is [HS16], p. 579 f., Proposition 3.1 (2). □

Corollary 3.5. *Let F/K be a finite discrete Galois module of order invertible in K with an extension $\mathscr{F}$ to a non-empty open subscheme U of C. Then there is an equality for all $r \geq 1$*

$$Ш^r(K, F) = \bigcap_{U' \subseteq U} \mathcal{D}^r(U', \mathscr{F}),$$

the intersection taken over all non-empty open subschemes U' of U.

Proof. From Proposition 3.4, we get

$$\operatorname{im}\left(\mathrm{H}^r_c(U', \mathscr{F}) \to \mathrm{H}^r(U', \mathscr{F})\right) = \ker\Big(\mathrm{H}^r(U', \mathscr{F}) \to \bigoplus_{v \in C \setminus U'} \mathrm{H}^r(K_v, \mathscr{F}_v)\Big).$$

Passing to the direct limit over all $U' \subseteq U$ non-empty open yields the result. (Note that $\varinjlim_{U'} \mathrm{H}^r(U', \mathscr{F}) = \mathrm{H}^r(K, F)$ by [Mil80], p. 88 f., Lemma III.1.16.) □

Corollary 3.6. *Let F/K be a finite discrete Galois module of order invertible in K with an extension $\mathscr{F}$ to a non-empty open subscheme U of C. There is an $U' \subseteq U$ open such that for all $U'' \subseteq U'$ non-empty open, one has*

$$\mathcal{D}^1(U'', \mathscr{F}) = Ш^1(K, F).$$

Proof. Note that the $\mathcal{D}^1(U'', \mathscr{F})$ are finite by Theorem 2.1. Hence the claim follows from Corollary 3.5 since a decreasing sequence of finite groups stabilizes. □

Theorem 3.7. *Let F/K be a finite discrete Galois module of order invertible in K. Then $Ш^1(K, F)$ is finite.*

Proof. By Proposition 2.3, there is an extension $\mathscr{F}$ of F to a non-empty open subscheme U of C. One has $Ш^1(K, F) \subseteq \mathcal{D}(U, \mathscr{F})$ by Corollary 3.5. But $\mathrm{H}^1_c(U, \mathscr{F})$ is finite by Theorem 2.1. □

Proposition 3.8. *Let F/K be a finite discrete Galois module of order invertible in K with an extension $\mathscr{F}$ to a non-empty open subscheme U of C. Then the sequence*

$$\bigoplus_{v \in |C|} \mathrm{H}^0(K_v, \mathscr{F}_v) \to \mathrm{H}^1_c(U, \mathscr{F}) \to \mathcal{D}^1(U, \mathscr{F}) \to 0$$

is exact.

Proof. The map

$$\bigoplus_{v \in |C|} \mathrm{H}^0(K_v, \mathscr{F}_v) \to \mathrm{H}^1_c(U, \mathscr{F}) \tag{3.1}$$

is constructed as follows: An element $\alpha \in \oplus_{v \in |C|} \mathrm{H}^0(K_v, \mathscr{F}_v)$ comes from the subgroup $\oplus_{v \in C \setminus V} \mathrm{H}^0(K_v, \mathscr{F}_v)$ for some $V \subseteq U$ open. By Proposition 3.4, one can map α to $\mathrm{H}^1_c(V, \mathscr{F})$, and this to $\mathrm{H}^1_c(U, \mathscr{F})$ by covariant functoriality of H^1_c. This does not depend on the choice of $V \subseteq U$ open since for $W \subseteq V$ open, the diagram

$$\begin{array}{ccc}
\bigoplus_{v \in C \setminus W} \mathrm{H}^0(K_v, \mathscr{F}_v) & \longrightarrow & \mathrm{H}^1_c(W, \mathscr{F}) \\
\uparrow & & \downarrow \\
\bigoplus_{v \in C \setminus V} \mathrm{H}^0(K_v, \mathscr{F}_v) & \longrightarrow & \mathrm{H}^1_c(V, \mathscr{F})
\end{array}$$

commutes.

That the sequence in Proposition 3.8 is a complex can be proved in the same way as in [HS16], p. 585, proof of Proposition 4.2.

The surjectivity of $\mathrm{H}^1_c(U,\mathscr{F}) \to \mathcal{D}^1(U,\mathscr{F})$ follows from the definition of $\mathcal{D}^1(U,\mathscr{F})$.

Let $V \subseteq U$ be open. By Proposition 3.4, there is a commutative diagram with exact upper row

$$\bigoplus_{v\in C\setminus V} \mathrm{H}^0(K_v,\mathscr{F}_v) \to \mathrm{H}^1_c(V,\mathscr{F}) \to \mathrm{H}^1(V,\mathscr{F}) \tag{3.2}$$

By Proposition 3.4 and as in [HS16], p. 586, proof of Proposition 4.2, there is a commutative diagram with exact upper row

$$\begin{array}{ccccc} \mathrm{H}^1_c(V,\mathscr{F}) & \longrightarrow & \mathrm{H}^1_c(U,\mathscr{F}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \\ & & \mathrm{H}^1(K,F) & \longrightarrow & \bigoplus_{v\in U\setminus V} \mathrm{H}^1(K_v,\mathscr{F}_v) \end{array} \tag{3.3}$$

Note that $\oplus_{v\in U\setminus V}\mathrm{H}^1(\kappa(v), i_v^*\mathscr{F}) = 0$ since $\kappa(v)$ is algebraically closed.

Now, let $\beta \in \ker(\mathrm{H}^1_c(U,\mathscr{F}) \to \mathcal{D}^1(U,\mathscr{F}))$. By exactness of the upper row in diagram (3.3), there is a $\gamma \in \mathrm{H}^1_c(V,\mathscr{F})$ mapping to β. Since the image of γ in $\mathrm{H}^1(K,F)$ is zero (because this holds for β), there is a $V \subseteq U$ non-empty open such that γ maps to 0 by $\mathrm{H}^1_c(V,\mathscr{F}) \to \mathrm{H}^1(V,\mathscr{F})$. By exactness of the upper row in (3.2), γ comes from $\bigoplus_{v\in|C|}\mathrm{H}^0(K_v,\mathscr{F}_v)$. □

Theorem 3.9. *Let F/K be a finite discrete Galois module of order invertible in K and $F' = \mathrm{Hom}(F,\mathbf{Q}/\mathbf{Z}(1))$. Then there is a perfect pairing of finite groups*

$$Ш^1(K,F) \times Ш^1(K,F') \to \mathbf{Q}/\mathbf{Z}.$$

The group $Ш^0(K,F)$ is finite and $Ш^r(K,F) = 0$ for $r > 1$.

Proof. That $Ш^0(K,F) \subseteq \mathrm{H}^0(K,F)$ is finite is trivial, and $Ш^r(K,F) = 0$ for $r > 1$ follows from $\mathrm{cd}(K) = 1$.

The groups $Ш^1$ are finite by Theorem 3.7. Define the groups $\mathcal{D}^1_{\mathrm{sh}}(U,\mathscr{F})$ for $U \subseteq C$ open by the exactness of the sequence

$$0 \to \mathcal{D}^1_{\mathrm{sh}}(U,\mathscr{F}) \to \mathrm{H}^1(U,\mathscr{F}) \to \prod_{v\in|C|} \mathrm{H}^1(K_v,\mathscr{F}_v). \tag{3.4}$$

These are finite by Theorem 2.2. Note that

$$Ш^1(K,F) = \varinjlim_{U'\subseteq U} \mathcal{D}^1_{\mathrm{sh}}(U',\mathscr{F}),$$

the direct limit taken over all $U' \subseteq U$ non-empty open.

By Corollary 3.6, there is a $U \subseteq C$ non-empty open such that for all $U' \subseteq U$ non-empty open, one has $\mathcal{D}^1(U',\mathscr{F}^\vee(1)) = Ш^1(K,F')$.

Taking the Pontryagin dual of the exact sequence in Proposition 3.8 for $\mathscr{F}^\vee(1)$ and using (3.4), one gets a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}^1_{\mathrm{sh}}(U',\mathscr{F}) & \longrightarrow & \mathrm{H}^1(U',\mathscr{F}) & \longrightarrow & \prod_{v\in|C|} \mathrm{H}^1(K_v,\mathscr{F}_v) \\ & & & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & \mathcal{D}^1(U',\mathscr{F}^\vee(1))^D & \longrightarrow & \mathrm{H}^1_c(U',\mathscr{F}^\vee(1))^D & \longrightarrow & \Big(\bigoplus_{v\in|C|} \mathrm{H}^0(K_v,\mathscr{F}_v^\vee(1))\Big)^D. \end{array}$$

Here, the middle vertical isomorphism is induced by Poincaré duality Theorem 2.2, and the right vertical isomorphism by local duality Proposition 3.1. Hence, passing to the colimit over all $U' \subseteq U$ non-empty open in this diagram yields the perfect duality pairing. □

4 Reductive groups

Proposition 4.1. *Let K be a field of cohomological dimension* 1*. Let T/K be a torus. Then one has* $\mathrm{H}^i(K,T)^{(p')} = 0$ *for* $i > 0$*.*

Proof. For $i > 2$, this follows from $\mathrm{scd}(K) = 2$ because of $\mathrm{cd}(K) = 1$, see [NSW08], p. 181, Exercise 1.

If $i = 2$, looking at the long exact cohomology sequence associated the Kummer sequence $0 \to T[n] \to T \to T \to 0$ for n invertible in K

$$0 = \mathrm{H}^2(K, T[n]) \to \mathrm{H}^2(K,T) \xrightarrow{n} \mathrm{H}^2(K,T) \to \mathrm{H}^3(K, T[n]) = 0$$

by $\mathrm{cd}(K) = 1$ gives us $\mathrm{H}^2(K,T)^{(p')} = 0$ since it is torsion.

If $i = 1$, note that $\mathrm{H}^1(K,T)$ is finite by [Izq17], Proposition 1.11. But $\mathrm{H}^1(K,T)$ is also divisible, so $\mathrm{H}^1(K,T) = 0$: The Kummer sequence $0 \to T[n] \to T \to T \to 0$ for n invertible in K gives an exact sequence

$$\mathrm{H}^1(K,T) \xrightarrow{n} \mathrm{H}^1(K,T) \to \mathrm{H}^2(K, T[n]) = 0,$$

the latter equality by $\mathrm{cd}(K) = 1$. □

Proposition 4.2. *Let K be of characteristic* 0*. Let G be a simply connected, quasisplit semisimple group over K not of type E_8. Then* $\mathrm{H}^1(K,G) = 0$*.*

Proof. First assume G absolutely almost simple. By [CPS12], p. 1027, Theorem 5.3, since K is of characteristic 0 and of cohomological dimension 1, one has an injection $\mathrm{H}^1(K,G) \hookrightarrow \mathrm{H}^3(K, \mathbf{Q}/\mathbf{Z}(2))$. But the latter group is trivial since K is of cohomological dimension 1.

In the general case, proceed as in [HS16], p. 598 f., proof of Proposition 6.2. □

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