Bayerische Kleine Arbeitsgemeinschaft

—

Towards Chromatic Homotopy Theory

The Landweber Exact Functor Theorem

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Introduction

This Bayerische Kleine AG presents a modern formulation of the *Landweber Exact Functor Theorem* (LEFT) of [Lan76] which stands at the intersection of algebraic topology (spectra and cohomology theories), algebraic geometry (moduli stacks, elliptic curves), and number theory/arithmetic geometry (formal group laws). The LEFT provides a general understanding of how ordinary cohomology theory, complex K-theory and Thom cobordism theory are related to isomorphism classes of formal group laws; it is a step towards the the potential description of the homotopy group of spheres via modular forms, *chromatic homotopy theory*.

In number theory, formal group laws appear in local class fields theory – Lubin-Tate theory [LT65] – and with elliptic curves over complete DVR's. Over complex manifolds, the first Chern classes of line bundles in singular cohomology and in complex K-theory provide two examples of additive and multiplicative formal group laws. The LEFT deals with the question *which formal group laws can be associated to a certain complex oriented cohomology theory*; It provides an explicit means of constructing cohomology theories and gives a dictionary between topology (spectra) and algebra (formal group laws).

This question relies on two moduli problems: the classification of formal group laws and formal groups over $\mathbb{Z}$, and the classification of complex oriented cohomology theories by spectra. Answers are respectively given by *the existence of a Lazard scheme* $\operatorname{Spec} L \simeq \operatorname{Spec} \mathbb{Z}[v_1, v_2, \dots]$ then a *fine moduli stack* $\mathcal{M}_{FG} = \lceil L/G^+ \rceil$ *classifying formal group laws and their isomorphisms*, and *by the universal Thom complex cobordism spectrum* MU. The LEFT then states that *any flat graded L-module* $\mathcal{F}$ *over* $\mathcal{M}_{FG}$ *provides a complex oriented homology theory of an E-spectrum given by* $E(-) = \mathrm{MU}(-) \otimes_L \mathcal{F}$*; such E-spectra are characterized among ring spectra by their even periodicity*. In this picture a key role is played by the stack structure via the flat stratification of $\mathcal{M}_{FG}$ by automorphisms, and by the explicit identification $L \simeq \mathrm{MU}_*$.

The goal of this Bayerische Kleine AG is to introduce the material required for the understanding of this result and of further results in chromatic homotopy theory.

Our main references are the lecture notes of J. Lurie [Lur10] and M. Hopkins [Hop99]. The reminders on cohomology theories, spectra, and the vector bundle constructions can be found in [AGP02]; advanced participants can consult [Ada74]. Speakers will provide explicit formulations of abstract results and will rely on examples and geometric motivations to remain accessible to algebraic topologists and geometers.

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PROGRAMME

Talk 1 – Formal Groups: Lazard ring & Classifications. (60 minutes)

"The moduli space of formal group laws is given by the Lazard scheme $\operatorname{Spec} L = \operatorname{Spec} \mathbb{Z}[a_1, a_2, \dots]$". Give the functorial definition of formal group laws and their isomorphisms with some examples ([Hop99], § 2, Def. 6.3 & Prop. 7.1), $\widehat{\mathbb{G}}_m \not\simeq \widehat{\mathbb{G}}_a$ over $\mathbb{F}_p$. Define the universal graded Lazard ring L, show that formal group laws are represented by L ([Rav04], Th. A2.1.8) and present Lazard's theorem ([Hop99], Th. 2.5). Give the classification of formal group laws both in characteristic 0 ([Lur10], § 2) – with examples – and over algebraically closed fields of positive characteristic p ([Hop99], Th. 13.8) after having introduced the concept of height via p-series $[p]_G$. Give the example of the formal group law for an elliptic curve ([Sil09], Cor. 7.5). If time remains, either illustrate the previous constructions in the case of the Lubin-Tate formal group ([Hop99], App. 13.10 f., § 14), or sketch the proof of the classification theorem via even graded indecomposables ([Lur10] § 12).

References: § 2 & 12–14 of [Lur10]; § 2, § 7 & § 13 of [Hop99]; Appendix A2 of [Rav04].

Talk 2 – Complex-oriented Cohomology Theories, Spectra & MU. (60 minutes)

"Complex-oriented cohomological functors are represented by a universal spectrum $\mathrm{MU}_* \simeq L$". Present for $X = \mathbb{C}P^\infty$ the properties that make the singular cohomology $H^*(X, \mathbb{Z})$ a complex-oriented cohomology theory (COCT) – multiplicative with a choice of Thom class (with short reminders on Chern classes & Thom spaces for vector bundles), [Lur10] § 4, [Hop99] § 1: *a COCT defines a formal group law*. Thom isomorphism motivates stable constructions: present the notion of spectra and state Adams-Brown's representability theorem for cohomological functors from the categories of spaces to graded abelian groups. Define the complex bordism spectrum $MU = \varinjlim MU(n)$ via Thom spaces and $U(n)$-vector bundles, show that this is the universal COCT – see below [Lur10] § 6 Rem. 11 and Th. 8. State Quillen's $L \simeq \pi_*\mathrm{MU}$ of [Lur10] § 7 Th. 1. Explain the result over $\mathbb{Q}$ via Hurewicz $L \to \pi_* MU \to H_*(MU; \mathbb{Z})$ and "2 orientations give isomorphic group laws" ([Lur10] § 7, or [Hop99] p. 20): for E an Eilenberg-MacLane spectrum $E \simeq \mathrm{MU} \wedge H\mathbb{Z}$ (skip the proof over $\mathbb{Z}$ via p-adic completion and Adams-Novikov spectral sequence).

References: § 4, § 6 & § 7 of [Lur10]; § 1 & § 4 of [Hop99]; § 8, § 11 & § 12 of [AGP02].

Talk 3 – The moduli stack $\mathcal{M}_{FG}$ of formal groups & the MU-construction. (60 minutes)

"There is a stack $\mathcal{M}_{FG}$ classifying formal groups, the obstruction to representablity by a spectrum is the non-constance of Iso." Present briefly the notion of a stack $\mathcal{M}$ (2-fiber product, the case of quotient stack $\mathcal{M} = [X/G]$) – [Hop99] § 8 – and a Hopf algebroid (A, Γ) – [Hop99] § 5: a Hopf algebroid (A, Γ) gives a stack $\mathcal{M}_{(A,\Gamma)}$. Present the case of $(A, \Gamma) = (\mathrm{MU}, \mathrm{MU}_*\mathrm{MU})$ which represents the formal group functor – [Hop99] § 6 & Prop. 6.5 (2), Talk 2 –, insist on the link between $\wedge$ and $StrictIso$: $\operatorname{Spec} \pi_*(\mathrm{MU} \wedge \mathrm{MU}) \simeq \operatorname{Spec} \mathbb{Z}[b_1, b_2, \dots] \times \operatorname{Spec} L$ ([Lur10], end § 10.) The moduli stack of formal groups $\mathcal{M}_{FG}$ is $\mathcal{M}_{(L,StrictIso)}$ ([Lur10], § 10) and has a stratification by heights $\mathcal{M}_{FG}^{\geqslant n}$: describe the strata over $\mathbb{Q}$ and $\mathbb{Z}_{(p)}$ ([Hop99] Def. 18.3, [Lur10], § 12 Cor. 3, § 13, Talk 1). Give the MU-construction $\mathrm{MU}_*(-) \otimes_L R$ – [Hop99] § 20 – and state *briefly* the goal of the LEFT in terms of η_1, η_2: $\operatorname{Spec} R_i \to \mathcal{M}_{FG}$ and the 2-obstruction in terms of change of orientation.

References: § 10 & 12 of [Lur10]; § 5, § 6, § 8 & § 20 of [Hop99]; Talks 1 & 2.

Talk 4 – Landweber Exact Functor Theorem & Flatness. (45 minutes)

"Landweber Exact Functor Theorem – Construction of COCT". Define flat quasi-coherent sheaves over $\mathcal{M}_{FG}$, [Lur10], § 15 Def. 1 & 3. Explain that the MU-construction gives a spectrum E for flat graded L-modules R – [Lur10], Cor. 6. State the link between flatness and StrictIso. – [Hop99] Th. 7.2 & Talk 3 –, give the regular criterion for flatness – [Lur10], § 15 Th. 9 – and the example of strata $\mathcal{M}_{FG}^{\geqslant n}$ – ibid. § 16 Claim 6. Present the examples of *Brown-Peterson homology* – [Lur10] § 15 Ex. 7 and *elliptic cohomology* – ibid. Ex. 13.

References: § 15 & § 16 of [Lur10]; [Hop99] p. 70; Talks 1 & 3.

Talk 5 – Landweber Exact Functor Theorem & Periodicity. (45 minutes)

"Landweber Exact Functor Theorem – Spectra". Define the notion of phantom maps and of a Landweber exact functor – [Lur10] § 17 – explain the relation between null-homotopic, evenly generated spectrum – ibid Th. 6 – and flatness – [Lur10] § 17 Cor. 9. Present the notion of even periodic spectrum ([Lur10] § 18 Def. 10) and state the last part of LEFT as in ibid. Prop. 11. *Application to complex K-theory:* $R = \mathbb{Z}[\beta, \beta^{-1}]$ gives a homology theory $E_*(-) := \mathrm{MU}(-) \otimes_L \mathbb{Z}[\beta, \beta^{-1}]$ isomorphic to complex K-theory with formal group law $f(x, y) = x + y + \beta xy$, ([Lur10] § 15 Ex. 12, [Hop99] p. 24); recover the Conner-Floyd isomorphism $KU \simeq \mathrm{MU} \otimes_L \mathbb{Z}[\beta, \beta^{-1}]$.

References: § 17 & § 18 of [Lur10]; Talks 2 & 4.

References

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