BAYERISCHE ARBEITSGEMEINSCHAFT

TOPOLOGICAL MODULAR FORMS

UNIVERSITÄT BAYREUTH

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The goal of this Bayerische Arbeitsgemeinschaft is to introduce interested (aspiring) Algebraic Geometers and Algebraic Topologists to the spectrum called TMF.

TMF stands for Topological Modular Forms, which are in some sense the "higher version" (in the sense of "Higher Algebra") of the classical ring of modular forms, which is an interesting object of study in Arithmetic Geometry. There, they arise as generalised functions on the moduli space of elliptic curves.

In the first talk, we will revise some fundamental notions of central importance such as formal group laws and their moduli stack $\mathcal{M}_{\mathrm{FG}}$.
After that, we will spend the second and third talk learning about the algebro-geometric theory of elliptic curves, their moduli problem and its solutions via different kinds of moduli schemes and stacks. In the third talk, we shall encounter modular forms from multiple perspectives, e.g. classically as certain functions on the upper-half plane and as sections of the sheaf $\omega^{\otimes k}$ on certain moduli spaces of elliptic curves.
In the fourth talk, the object TMF shall finally be introduced. Its construction is difficult and relies on $\mathcal{O}^{top}$, a sheaf of E_∞-ring spectra whose existence relies on a theorem of Goerss-Hopkins-Miller.
Finally, in the fifth talk we will try to compute (some of) the homotopy groups of TMF using tools such as the descent and the Adams-Novikov spectral sequence.

All talks should last at most one hour.

Talk 1: Prelude

Set the scene by giving an overview over the following things from the last Bayerische AG:

- Formal group laws and formal groups [Mei19, §2.7].
- Stacks on sites, the étale and fpqc site of a scheme (or stack) and the moduli stack $\mathcal{M}_{FG}$ of formal groups [Mei19, §§4.3–4.5].
- The Landweber Exact Functor Theorem, both the "down-to-earth" [Mei19, §4.1] and the "stacky" version [Mei19, §4.7].

Talk 2: Elliptic Curves

- Explain the notion of an elliptic curve over an arbitrary base scheme S and sketch why it carries a unique group scheme structure [KM85, (2.1)]. Explain and motivate the notion of a generalized elliptic curve [DR73].
- Explain why this is equivalent to the classical definition: Construct the Weierstraß equation from the choice of a formal parameter at 0 [KM85, (2.2)]. Explain why elliptic curves over $\mathbb{C}$ are equivalent to complex tori [DS05, 1.4].

- Construct the universal elliptic curve over a scheme with 2 and/or 3 invertible [KM85, (2.2.6), (2.2.8), (2.2.10)]; [Del75, Proposition 2.5, 3.5].
- State that the N-multiplication morphism is finite and flat of degree N^2 with kernel étale locally isomorphic to $(\mathbb{Z}/N)^2$ if N is invertible on S [KM85, (2.3)]. (We need this for the next talk.)

Talk 3: Moduli Spaces of Elliptic Curves and Modular Forms

- Introduce moduli problems of elliptic curves as (a) the affine smooth moduli curves $Y(N)$ and $Y_1(N)$ over $\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ and their smooth compactifications $X(N)$ and $X_1(N)$ [KM85]; [DR73, IV, 2.2, 2.9], (b) the moduli stacks $\mathcal{M}(N) \to \mathcal{M}_{\mathrm{ell}}$ and $\mathcal{M}_1(N) \to \mathcal{M}_{\mathrm{ell}}$ and when they are schemes [Mei19, Proposition 3.25, Theorem 4.35 and §4.8].
- Explain the morphism $\mathcal{M}_{\mathrm{ell}} \to \mathcal{M}_{\mathrm{FG}}$ from the last Bayerische AG [Mei19, Theorem 4.57].
- Introduce (weakly) modular forms (of weight k and with Γ-level structure) in three a priori different ways: (a) classically as certain functions on the upper half plane and homogeneous functions on lattices [DS05], (b) as sections of line bundles on $X(N)$, $X_1(N)$ [DR73, IV], (c) as sections of the sheaf $\omega^{\otimes k}$ on $\mathcal{M}_{\mathrm{ell}}$ with their basic geometric properties [KM85, (8.1.7.1)]; [MO18, Appendix A]. Explain the relation between these definitions and what are their advantages.
- Compute the ring of modular forms over $\mathbb{Z}[\frac{1}{6}]$ [DFHH14, Theorem 3.8] and $\mathbb{Z}$ [Del75, Proposition 6.1][1].

Talk 4: Topological Modular Forms

- Explain the definition of $\mathcal{O}^{hom}$, a presheaf of homology theories on the category of flat morphisms $[\operatorname{Spec} R/\mathbb{G}_m] \to \mathcal{M}_{ell}$, where $[\operatorname{Spec} R/\mathbb{G}_m]$ is the quotient (stack) of an affine scheme by a $\mathbb{G}_m$-action as described in [Mei19, Exercise 4.67]. Use $\mathcal{O}^{hom}$ to define $TMF[\frac{1}{6}]$ and show that $\pi_{2k}TMF[\frac{1}{6}] \cong MF_{k,\mathbb{Z}[\frac{1}{6}]}$ [Mei19, §4.8].
- State the theorem of Goerss-Hopkins-Miller [Mei19, Theorem 5.1][2], which lifts the presheaf of homology theories $\mathcal{O}^{hom}$ to a presheaf of E_∞-ring spectra $\mathcal{O}^{top}$. Elaborate on what it means to be a presheaf of (ring) spectra[3] [DFHH14, Chapter 5, §2.3].
- Give the definition of TMF as "global sections of $\mathcal{O}^{top}$" [Mei19, Definition 5.2]. Explain what the homotopy groups $\pi_k(\mathcal{F})$ of a presheaf of spectra are. Give [Mei19, Proposition 5.7] which says in particular that the new definition of TMF coincides with the one from the beginning of the talk when 6 is invertible.

Talk 5: Computation of $\pi_*(TMF)$ via the Descent and Adams-Novikov Spectral Sequences

- Introduce the Descent spectral sequence for TMF [DFHH14, Chapter 5, §3]; [Mei19, §5.2–5.4].
- Introduce the Adams and Adams-Novikov spectral sequences and explain how it could be used to calculate $\pi_*(\mathbb{S})$. In the case of TMF, discuss the comparison map to the Descent spectral sequence [Mei19, §5.5].
- Give (or rather sketch) the computation of the 3-local homotopy groups of TMF [Mei19, §5.8].

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[1] Might be sketched as this is more difficult.

[2] Do not say much about its proof unless you have extra time in the end.

[3] The most modern way to approach this is by using the ∞-category Sp of spectra, although the reference [Mei19] uses Orthogonal Spectra. Coordinate your choice of setup with the speaker for Talk 5.

References

[Del75] P. Deligne, *Courbes elliptiques: formulaire d'après J. Tate*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1975, pp. 53–73. Lecture Notes in Math., Vol. 476 (French).

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 143–316. Lecture Notes in Math., Vol. 349 (French).

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005.

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014.

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.

[MO18] Lennart Meier and Viktoriya Ozornova, *Rings of modular forms and a splitting of* $TMF_0(7)$, 2018. preprint, https://arxiv.org/abs/1812.04425.

[Mei19] Lennart Meier, *Elliptic Homology and Topological Modular Forms*, 2019. lecture notes, http://www.staff.science.uu.nl/~meier007/TMF-Lecture.pdf.