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KLEINE AG MAY 2020: WEIL CONJECTURES

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André Weil in 1949 formulated a very influential set of conjectures on the Hasse–Weil zeta-function ζ_X of a smooth projective variety X over a finite field, predicting that ζ_X is rational, satisfies a functional equation, has zeros and poles of a certain absolute value ("Riemann Hypothesis"), and knows the Betti numbers if X arises from a variety of good reduction over a number field. Since ζ_X can be described as a generating series by counting numbers of points on the variety in field extensions of the finite ground field, this link with topology is rather surprising.

The Weil conjectures where proved over the course of the next thirty years with beautiful methods that were itself very influential in the development of arithmetic geometry. The proof of the first part was accomplished by Grothendieck ('65, together with Artin, Verdier, ...) with the development of ℓ-adic étale cohomology. Deligne ('74, '80) subsequently proved the remaining, most difficult, part of the Weil conjectures: The "Riemann hypothesis".

In this one-day workshop, we follow Deligne's second proof of the Riemann hypothesis part of the Weil conjecture [Del80], with Laumon's simplification via the ℓ-adic Fourier transform [Lau87].

The abstracts of the talks are as follows (a more detailed version can be found on the website).

1. Talk – Introduction: The Weil Conjectures and cohomological interpretation of L-functions

This talk gives an overview over the Weil conjectures, and briefly discusses how ℓ-adic cohomology can be used to prove the first three statements of the Weil conjectures via the cohomological interpretation of L-functions.

Duration: 60 minutes
Material: §1 of [Del74] *and* [Del80, Introduction].

2. Talk – Weights of étale sheaves

This talk introduces some of the basic language used in Deligne's paper: lisse $\mathbb{Q}_l$-modules and their ℓ-adic cohomology, the Frobenius action, weights, pure and mixed sheaves, and the Grothendieck–Lefschetz formula. We then formulate the Main Theorem of [Del80] in its general form, and finally discuss the Semicontinuity Theorem, a first step towards the proof of the former.

Duration. 60 minutes
Material: [KW01, I.2] *and* [Del80, 1.2, I.8.9].

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3. Talk – Real sheaves and Rankin's Trick

This talk begins by sketching the proof of the first simple instance of the Main Theorem, namely sheaves of rank 1 on a curve, by using that the monodromy in this case is easy to control. This motivates the introduction of determinant weights.

We then introduce real sheaves, and use "Rankin's Trick" to prove that real sheaves are mixed, a second important ingredient in the proof of the Main Theorem.

Duration: 60 minutes
Material: [KW01, I.2-1.4] *and* [Del80, 1.5].

4. Talk – Proof of Main Theorem: Reduction to the case of curves

This talk begins the poof of the Main Theorem by reducing to the case of curves. We then deal with the cases of H^0 and H^2 as well as special cases like geometrically constant sheaves, and make some further reductions to simplify the situation (namely to [KW01, Claim 6.3]).

We then discuss a few basics about Grothendieck's function–sheaf correspondence as a preparation for the next talk. *Duration: 60 minutes*
Material: [KW01, I.5-7] *and* [Del80, 3.3].

5. Talk – Fourier theory

To complete the proof of the Weil Conjectures following Laumon, this talk introduces the ℓ-adic Fourier transform and explains its relation to L-functions. We then discuss Plancherel's Formula, and use it to prove the Main Theorem.

Duration: 60 minutes
Material: [KW01, I.2,5,6].

References

[Del74] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[Del80] Pierre Deligne. La conjecture de Weil II. *Pubications mathematiques de l'I.H.E.S.*, 52:137–252, 1980.

[KW01] Reinhardt Kiehl and Rainer Weissauer. *Weil conjectures, perverse sheaves and l'adic Fourier transform*, volume 42 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2001.

[Lau87] Gérard Laumon. Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil. *Pubications mathematiques de l'I.H.E.S.*, (65):131–210, 1987.