

EFFICIENT IRREDUCIBILITY OF RESIDUAL GALOIS REPRESENTATIONS OF ABELIAN SURFACES WITH RM OVER \mathbb{Q}

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Setup and basic properties

- Let A be a principally polarized abelian *surface* over \mathbb{Q} with RM by \mathcal{O} over \mathbb{Q} and $\bar{\mathbb{Q}}$.
- A is *absolutely simple* and hence \mathbb{Q} -isomorphic to $\text{Jac}_{C/\mathbb{Q}}$ for a unique genus-2 curve C (Torelli + $\dim \mathcal{M}_2 = 3 = \dim \mathcal{A}_{2,1,1}$).
- A is of GL_2 -type over \mathbb{Q} and hence *modular* of some level N (Ribet + Serre's modularity conjecture).

Computationally feasible effective irreducibility of $\rho_{\mathfrak{p}}$

Question: Which $\rho_{\mathfrak{p}} : G_{\mathbb{Q}} \rightarrow \text{Aut}(A[\mathfrak{p}])$ are irreducible for $\mathfrak{p} \mid p$ a finite prime of \mathcal{O} ?

Theorem [Rib76]: All but finitely many $\rho_{\mathfrak{p}}$ are absolutely irreducible. Disadvantage: *ineffective*

Theorem [DL14]: Effective, but *computationally infeasible* upper bound on $|\mathfrak{p}|$ depending on the stable Faltings height of A .

Goal

For a *concretely* given A of **dimension 2**, *efficiently* determine the p with $\rho_{\mathfrak{p}}$ irreducible for all $\mathfrak{p} \mid p$.

- **Adapt Dieulefait's algorithm [Die02]** for $\text{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ to real multiplication by \mathcal{O} , distinguish by decomposition type of p in \mathcal{O} : Either p is inert or it all primes above it have degree 1.
- $A[p]$ is a free $\mathcal{O}/p[\text{Gal}_{\mathbb{Q}}]$ -module of rank 2.
- Need to know the characteristic polynomials of the Frobenii at $\ell \neq p$ acting on $A[p]$: Local zeta function of the associated curve C .
- Raynaud's theorem [Ray74] also holds for $p \parallel N$ of semiabelian reduction! The toric part is unramified.

Theorem (square-free conductor)

Assume that the conductor N of the strictly compatible system of \mathfrak{p} -adic Galois representations $(\rho_{\mathfrak{p}})_{\mathfrak{p}|p}$ is *square-free*.

If p is a prime for which the localization $\mathcal{O}_{\mathfrak{p}}$ is a DVR for all $\mathfrak{p} \mid p$, $\rho_{\mathfrak{p}}$ is reducible for some \mathfrak{p} if and only if $A'(\mathbb{Q})[p] \neq 0$ for some A' \mathbb{Q} -isogenous to A (via an isogeny with kernel contained in $A[\mathfrak{p}]$).

- For a prime $2 < \ell \neq p$ of good reduction, $A'(\mathbb{Q})[p] \mid \#A'(\mathbb{F}_{\ell}) = \#A(\mathbb{F}_{\ell})$, so $A'(\mathbb{Q})[p] \neq 0$ can be checked without actually knowing A' . This gives effectively finitely many \mathfrak{p} with $\rho_{\mathfrak{p}}$ reducible.
- We can also algorithmically treat N not square-free, but with less precise results.

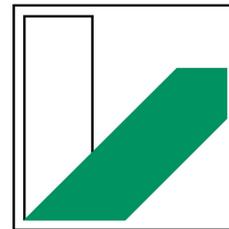
References

[Die02] Luis V. Dieulefait, *Explicit determination of the images of the Galois representations attached to abelian surfaces with $\text{End}(A) = \mathbb{Z}$* , Experiment. Math. **11** (2002), no. 4, 503–512 (2003).

[DL14] Davide Lombardo, *Explicit surjectivity of Galois representations attached to abelian surfaces and GL_2 -varieties*, 2014. Preprint, arXiv:1411.1703.

[Ray74] Michel Raynaud, *Schémas en groupes de type (p, \dots, p)* , Bull. Soc. Math. France **102** (1974), 241–280 (French).

[Rib76] Kenneth A. Ribet, *Galois action on division points of Abelian varieties with real multiplications*, Amer. J. Math. **98** (1976), no. 3, 751–804.



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Semistable example:

$A = \text{Jac}(X_0(35)/\langle w_7 \rangle)$ (**easier**)

A is the 2-dimensional isogeny factor of $J_0(35)$. It is semistable and absolutely simple with real multiplication by the ring of integers \mathcal{O} of $\mathbb{Q}(\sqrt{17})$ over \mathbb{Q} .

Result

$A(\mathbb{Q})[2] = \mathbb{Z}/2$ and $\rho_{\mathfrak{p}}$ is reducible exactly for one of the primes over 2 of \mathcal{O} .

Corollary: Since \mathcal{O} is a PID, A has no cyclic isogenies of prime power degree other than endomorphisms, except for the prime 2.

Non-semistable example:

$A = \text{Jac}(X_0(125)/\langle w_5 \rangle)$ (**more difficult**)

A is absolutely simple with real multiplication by the ring of integers \mathcal{O} of $\mathbb{Q}(\sqrt{5})$ over \mathbb{Q} , but *not* semistable.

Result

$A(\mathbb{Q})_{\text{tors}} = 0$ and $\rho_{\mathfrak{p}}$ is irreducible except perhaps for one of the primes over 3 (good reduction) and 5 (split multiplicative reduction) of \mathcal{O} .

Corollary: Since \mathcal{O} is a PID, A has no cyclic isogenies of prime power degree other than endomorphisms, except perhaps for the primes 3 and 5.

Computational problem

Either find a means to compute these extra isogenies or prove that they cannot exist.

Calculations on the *Kummer surface* show: For $p = 3$ inert in \mathcal{O} , ρ_3 is irreducible.