On an analogue of the conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher dimensional bases over finite fields

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September 19, 2018

Abstract
We formulate an analogue of the conjecture of Birch and Swinnerton-Dyer for Abelian schemes with everywhere good reduction over higher dimensional bases over finite fields. We prove some conditional results for the $p'$-part on it, and prove the $p'$-part of the conjecture for constant or isogenous Abelian schemes, in particular the $p'$-part for (1) relative elliptic curves with good reduction or (2) Abelian schemes with constant isomorphism type of $A[p]$ or (3) Abelian schemes with supersingular generic fibre over products of curves, Abelian varieties and $K_3$ surfaces, and the full conjecture for relative elliptic curves with good reduction over curves and for constant Abelian schemes over arbitrary bases. We also reduce the conjecture to the case of surfaces as the basis.

Keywords: $L$-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture; Heights; Étale cohomology, higher regulators, zeta and $L$-functions; Abelian varieties of dimension $>1$; Étale and other Grothendieck topologies and cohomologies; Arithmetic ground fields

MSC 2010: 11G40, 11G50, 19F27, 11G10, 14F20, 14K15

1 Introduction
If $K$ is a global field, i.e. a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q(t)$, the conjecture of Birch and Swinnerton-Dyer for an Abelian variety $A/K$ relates global invariants, like the rank of the Mordell-Weil group $A(K)$, the order of the Tate-Shafarevich group $\text{III}(A/K)$ (a group measuring the failure of the Hasse principle for principal homogeneous spaces of $A/K$) and the determinant of the height pairing $A(K) \times A'(K) \to \mathbb{R}$ with $A'$ the dual Abelian variety to the vanishing order of the $L$-function $L(A/K, s)$ (built up from the number of points of the reduction of $A$ at the primes of $K$) at $s = 1$ and the special $L$-value at this point. The aim of this article is to extend this setting from the classical situation of a curve $C$ over a finite field to the case of a higher dimensional basis over finite fields.

Even for elliptic curves over the rationals, this is a difficult problem. The function field case is more accessible since the situation is more geometric as one has a ground field the algebraic closure of which one can pass to, but up to now, there have been only (mostly conditional) results over curves over finite fields: For Abelian varieties over global function fields, John Tate [Tat66] considered the problem for Jacobians of curves, and the first result is due to James Milne [Mil68]: He proved the conjecture of Birch and Swinnerton-Dyer for constant Abelian schemes over global function fields, i.e. Abelian schemes of the form $A = A \times_k X$ with $A/k$ an Abelian variety over a finite field $k$ and $X/k$ a smooth projective geometrically connected curve. Later, Peter Schneider [Sch82] proved a conditional result for Abelian varieties over global function fields, namely that the $p'$-part of the conjecture of Birch and Swinnerton-Dyer ($p$ the characteristic of the ground field) holds if for one $\ell \neq p$, the $\ell$-primary part of the Tate-Shafarevich group is finite. In [Bad02], Werner Bauer proved an analogue of Schneider’s result for the $p'$-part of the conjecture, but only for Abelian varieties with good reduction; finally, Kazuya Kato and Fabien Trihan [KT03] extended Bauer’s result to the case of bad reduction. Tate and Shafarevich [TS67] gave examples of elliptic curves over $\mathbb{F}_q(t)$ of arbitrarily large rank and Douglas Ulmer [Ulm02] proved the conjecture for certain non-isogenous elliptic curves over $\mathbb{F}_q(t)$ with arbitrarily large rank.

In section 2 we proceed by generalising Schneider’s arguments to the case of a higher dimensional basis $X$ over a finite field $k$. A key point was to find the correct definition of the $L$-function in the higher dimensional
setting. Let $\mathcal{A}/X$ be an Abelian scheme. The Kummer sequence for $\mathcal{A}/X$ on the étale site of $X$ induces a short exact sequence

$$0 \to \mathcal{A}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^1(X, T_\mathcal{A}) \to T_\mathcal{A} \pi_1(\mathcal{A}/X) \to 0$$

with $H^1(X, T_\mathcal{A}) = \lim_{\leftarrow n} H^1(X, \mathcal{A}^{[n]})$. Since $\pi_1(\mathcal{A}/X)[\ell^{\infty}]$ is cofinitely generated, $T_\mathcal{A} \pi_1(\mathcal{A}/X) = 0$ iff $\pi_1(\mathcal{A}/X)[\ell^{\infty}]$ is finite. This gives us the link between the algebraic rank $\text{rk}_Z \mathcal{A}(X)$ and $H^1(X, T_\mathcal{A})$. Using the Hochschild-Serre spectral sequence $H^p(G_k, H^q(X, T_\mathcal{A})) \Rightarrow H^{p+q}(X, T_\mathcal{A})$, one relates $H^1(X, T_\mathcal{A})$ to $H^1(X, T_\mathcal{A})^{G_k}$. Then one uses Lemma 2.25 to relate the vanishing of the $L$-function to the algebraic rank and the special $L$-value at $s = 1$ to orders of cohomology groups and determinants of cohomological pairings. The proof is complicated by the fact that one has more non-vanishing cohomology groups than in the case of a curve as a basis. For example, setting $d = \dim X$, if $d = 1$, Poincaré duality is a pairing between $H^1(X, \mathcal{F}) \times H^2_{\text{et}}(X, \mathcal{F}^\vee(d)) \to \mathbb{Z}_p$, whereas for general $d > 1$, it is a pairing $H^1(X, \mathcal{F}) \times H^{2d-1}_{\text{et}}(X, \mathcal{F}^\vee(d)) \to \mathbb{Z}_p$.

In section 3 we study two cohomological pairings given by cup product in cohomology:

$$\langle \cdot, \cdot \rangle_t : H^1(X, T_\mathcal{A})_{\text{tors}} \times H^{2d-1}(X, T_\mathcal{A}(\mathcal{A}^t)(d-1))_{\text{tors}} \to H^{2d}(X, \mathcal{Z}_p(d)) \to H^d(\mathcal{X}, \mathcal{Z}_p(d)) = \mathbb{Z}_p$$

If one $\ell$-primary component of the Tate-Shafarevich group of $\mathcal{A}/X$ is finite, we relate the pairing $\langle \cdot, \cdot \rangle_t$ to the Néron-Tate height pairing, and show that the determinant of the pairing $\langle \cdot, \cdot \rangle_t$ equals $1$. This is done by generalising Schneider’s arguments comparing $\langle \cdot, \cdot \rangle_t$ with Bloch’s height pairing from [Blo80]. Again, the higher dimensional case is more involved.

In section 4 we specialise to the case of an isocapeutic Abelian scheme, and deduce in section 5 from a descent theorem of our previous article [Kel10], p. 238, Theorem 4.29 the conjecture of Birch and Swinnerton-Dyer for relative elliptic curves or Abelian schemes with constant isomorphism type of $\mathcal{A}^t[p]$ over products of curves and Abelian varieties by showing these are isocompatible since the moduli scheme $Y(N)$ is affine for $N \geq 3$ resp. since the Ekedahl-Oort stratification is quasi-affine. We also prove the conjecture for supersingular Abelian schemes.

In section 6 we reduce the conjecture to the case of a surface (and in special cases also of a curve) as a basis using Poonen’s Bertini theorem for varieties over finite fields.

Our main results are as follows:

In section 2, we first introduce a suitable $L$-function $L(\mathcal{A}/X, s)$ for Abelian schemes $\mathcal{A}$ over a smooth projective base scheme $X$ over a finite field of characteristic $p$ (see Remark 2.8 for a motivation):

$$L(\mathcal{A}/X, s) = \frac{\det(1 - t \text{Frob}_q^{-1} | H^1(X, V_\mathcal{A}^t))}{\det(1 - t \text{Frob}_q^{-1} | H^0(X, V_\mathcal{A}^t))}$$

We then prove that an analogue of the conjecture of Birch and Swinnerton-Dyer holds for the prime-to-$p$ part, with two cohomological pairings $\langle \cdot, \cdot \rangle_t$ and $\langle \cdot, \cdot \rangle_r$ in place of the height pairing, provided that for one $\ell \neq p$ the $\ell$-primary component of the Tate-Shafarevich group $\pi_1(\mathcal{A}/X) := H^1_{\text{et}}(X, \mathcal{A})$ is finite or, equivalently, if the analytic rank equals the algebraic rank.

The Tate-Shafarevich group is studied in a previous article [Kel16], section 4, especially Theorem 4.4 and 4.5. There, we show:

$$\pi_1(\mathcal{A}/X) = \ker \left( H^1(K, \mathcal{A}) \to \prod_{x \in S} H^1(K_{x, \mathcal{A}}^\text{nr}, \mathcal{A}) \right),$$

where $K_{x, \mathcal{A}}^\text{nr} = \text{Quot}(\mathcal{O}_{X,x}^\text{sh})$, and $S$ is either (a) the set of all points of $X$, or (b) the set of all points of $X$, or (c) the set $X^{(1)}$ consisting of codimension-1 points of $X$, and $\mathcal{A} = \text{Pic}_{g_{X^t}/X}$ for a relative curve $\mathcal{C}/X$ with everywhere good reduction admitting a section, and $X$ is a variety over a finitely generated field. Here, one can replace $K_{x, \mathcal{A}}^\text{nr}$ by $K_{x, \mathcal{A}}^\text{nr}$ if $\kappa(x)$ is finite, and $K_{x, \mathcal{A}}^\text{nr}$ by $\text{Quot}(\mathcal{O}_{X,x}^\text{sh})$ and $\text{Quot}(\mathcal{O}_{X,x}^\text{sh})$, respectively, if $x \in X^{(1)}$.

More precisely, we get the following first main result (see Theorem 2.44):

**Theorem 1.** Let $X/k$ be a smooth projective, geometrically connected variety over a finite field $k = \mathbb{F}_q$ and $\mathcal{A}/X$ an Abelian scheme. Set $\bar{X} = X \times_k \bar{k}$ and let $\ell \neq \text{char } k$ be a prime. Let $\rho$ be the vanishing order of $L(\mathcal{A}/X, s)$ at $s = 1$ and define the leading coefficient $c = L^*(\mathcal{A}/X, 1)$ of $L(\mathcal{A}/X, s)$ at $s = 1$ by

$$L(\mathcal{A}/X, s) \sim c \cdot (1 - q^{-s})^\rho \sim c \cdot (\log q)^\rho (s - 1)^\rho \quad \text{for } s \to 1.$$

Then one has $\rho \geq \text{rk}_Z \mathcal{A}(X)$, and the following statement is equivalent:

(a) $\rho = \text{rk}_Z \mathcal{A}(X)$
(b) \( III(\mathcal{O}/X)[T]\) is finite
If these hold, one has for all \( \ell \neq \text{char } k \) the equality for the leading Taylor coefficient
\[
|c|_{\ell}^{-1} = \frac{\|III(\mathcal{O}/X)[T]\| \cdot R_{\ell}(\mathcal{O}/X)}{|\mathcal{O}(X)[T]\times| \cdot |H^2(X, T_{\sigma})|}
\]
and the prime-to-\( p \) part of the Tate-Shafarevich group \( III(\mathcal{O}/X)[\text{non-p}] \) is finite. Here \( \mathcal{O}(X) = A(K) \) with \( A \) the
generic fibre of \( \mathcal{O}/X \) and \( K = k(X) \) the function field of \( X \), and the regulator \( R_{\ell}(\mathcal{O}/X) \) is the determinant of
the cohomological pairing \( \langle , \rangle_{\ell} \) divided by the determinant of the cohomological pairing \( \langle , \rangle_{\ell} \).

For example, (a) holds if \( L(\mathcal{O}/X, 1) \neq 0 \) (Remark 2.45(a)), and (b) holds under mild conditions if \( \mathcal{O}/X \) is
isocentral (Theorem 4.16, Remark 4.17, and Theorem 5.10).

In section 5, we construct a higher-dimensional analogue
of the Néron-Tate canonical height pairing with \( A' \) the dual Abelian variety, and show the second main result, which identifies the cohomological regulator \( R_{\ell}(\mathcal{O}/X) \) in Theorem 1 with a geometric one:

**Theorem 2.** Let \( \ell \) be a prime different from char \( k \). Assume that \( III(\mathcal{O}/X)[T]\) is finite.
(a) The Néron-Tate canonical height pairing \( \langle , \rangle \) gives the pairing \( \langle , \rangle_{\ell} \) after tensoring with \( Z_\ell \) up to a known
factor, the integral hard Lefschetz defect, see Definition 5.6.
(b) The cohomological pairing \( \langle , \rangle_{\ell} \) has determinant 1.

More precisely, the pairing \( \langle , \rangle \) depends on the choice of a very ample line bundle on \( X \), but the comparison isomorphism also, and the two choices cancel each other out; see Remark 3.31. For (a), see Theorem 3.11 and Theorem 3.35 for (b). In Theorem 3.11, we identify the cohomological pairing \( \langle , \rangle_{\ell} \) with a trace pairing in the case of \( \mathcal{O}/X \) a constant Abelian variety, and in Theorem 4.15 with another pairing if \( X \) is a curve.

We prove the conjecture of Birch and Swinnerton-Dyer for constant Abelian schemes, see 4.30

**Theorem 3.** Let \( X/k \) be a smooth projective geometrically connected variety over a finite field \( k = \mathbf{F}_q \) and \( B/k \) an
Abelian variety of dimension \( d \). Set \( \tilde{X} = X \times_k \bar{k} \) and \( \mathcal{O} = B \times_k X \), and let \( K = k(X) \) be the function field of \( X \).
The \( L \)-function of \( \mathcal{O}/X \) is defined in Definition 4.23. Assume
(a) the Néron-Severi group of \( \tilde{X} \) is torsion-free and
(b) the dimension of \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \) as a vector space over \( k \) equals the dimension \( g \) of the Albanese variety of \( \tilde{X}/k \).
Then:
1. The Tate-Shafarevich group \( III(\mathcal{O}/X) \) is finite.
2. The vanishing order equals the Mordell-Weil rank \( r \) : \( \text{ord}_s = 1 L(\mathcal{O}/X, s) = r \mathcal{O}(X) = r \text{rk } A(K). \)
3. There is the equality for the leading Taylor coefficient
\[
L^*(\mathcal{O}/X, 1) = q^{(q-1)d} (\log q)^{\frac{3}{2}} |\mathcal{O}(X)[T]| R_{\ell}(\mathcal{O}/X) |\mathcal{O}(X)[T]|.
\]
Here, \( R(\mathcal{O}/X) \) is the determinant of the trace pairing \( \text{Hom}(A, B) \times \text{Hom}(B, A) \to \text{End}(A) \times \mathbf{Z} \) with \( A \) the
Albanese variety of \( X \), or, see Theorem 4.11, the determinant of a cohomological pairing, and, if \( X \) is a curve, the
detector of another pairing or the Néron-Tate canonical height pairing, see 4.15.

Combining the finiteness of \( III(\mathcal{O}/X) \) for constant \( \mathcal{O}/X \) [Mil68], p. 98, Theorem 2 and the descent of finiteness of
\( III \) under \( \ell \)-alterations [Kel10], p. 238, Theorem 4.29, we obtain (see Theorem 4.16 and Theorem 5.10):

**Theorem 4.** Let \( X/k \) be a smooth projective geometrically connected variety over a finite field \( k = \mathbf{F}_q \) and \( \mathcal{O}/X \) an isocentral Abelian scheme, i. e. such that there exists a proper, surjective, generically étale morphism \( f : X' \to X \) such that \( f^* \mathcal{O} : = \mathcal{O} \times_X X' \) is constant. Assume that (a) the Néron-Severi group of \( X' \) is
torsion-free and (b) the dimension of \( H^1(X', \mathcal{O}_{X'}) \) as a vector space over \( k \) equals the dimension of the Albanese
variety of \( X'/k \). Then the \( p' \)-part of the conjecture of Birch and Swinnerton-Dyer holds for \( \mathcal{O}/X \).

Note that we do not need \( f \) to be of general degree prime to \( \ell \) since \( \mathcal{O}/X \) is \( \ell \)-isocentral (isocentral for a
generically étale morphism \( f : X' \to X \) of general degree prime to \( \ell \) for some \( \ell \), and then we can use (a) \( \implies \)
(b) from Theorem 2.44 to get independence from \( \ell \). This also extends the known, classical results for Abelian
varieties over one-dimensional global function fields, where the constant case had been settled by Milne [Mil68],
p. 100, Theorem 3. In Theorem 5.6, we prove that relative elliptic curves are isocentral. In Corollary 5.11 and
Corollary 5.12, we conclude:
Theorem 5. The $p'$-part of the conjecture of Birch and Swinnerton-Dyer holds for (1) relative elliptic curves or (2) Abelian schemes with constant isomorphism type of $A[p]$ or (3) Abelian schemes with supersingular generic fibre over a product of curves, Abelian varieties and $K3$ surfaces over a finite field, and the full conjecture for elliptic curves with good reduction over $1$-dimensional global function fields.

The prime-to-$p$ part of the Brauer group of a relative elliptic curve $E$ over a product of curves, Abelian varieties and $K3$ surfaces is finite, the Tate conjecture holds in dimension 1, and the Brauer group of a relative elliptic curve $E$ over a smooth proper curve $C$ over a finite field is finite of square order and equals $\mathbb{I}(E/C)$, and the Tate conjecture holds for $E$.

In section $6$ we prove:

Theorem 6. If the analogue of the conjecture of Birch-Swinnerton-Dyer holds for all Abelian schemes over all smooth projective geometrically integral surfaces, it holds over arbitrary dimensional bases.

More precisely, if there is a sequence $S \hookrightarrow \ldots \hookrightarrow X$ of ample smooth projective geometrically integral hypersurface sections with a surface $S$ and the conjecture holds for $A/S$, it holds for $A/X$.

If there is a smooth projective ample geometrically integral curve $C \hookrightarrow S$ with $\text{rk} A(S) = \text{rk} A(C)$, the analogue of the conjecture of Birch and Swinnerton-Dyer for $A/S$ is equivalent to the conjecture for $A/C$.

Notation. For an Abelian group $A$, let $A_{\text{tors}}$ be the torsion subgroup of $A$, and $A_{n\text{-tors}} = A/A_{\text{tors}}$. Let $A_{\text{div}}$ be the maximal divisible subgroup of $A$ and $A_{n\text{-div}} = A/A_{\text{div}}$. Denote the cokernel of $A \to A$ by $A/n$ and its kernel by $A[n]$, and the $p$-primary subgroup $\lim\limits_{\to} A[p^n]$ by $A[p]\infty$. Canonical isomorphisms are often denoted by "$\sim$". If not stated otherwise, all cohomology groups are taken with respect to the étale topology. We denote Pontryagin duality by $(\cdot)^d$, duals of $R$-modules or $\ell$-adic sheaves by $(\cdot)^\vee$, and duals of Abelian schemes and Cartier duals by $(\cdot)^t$. The $\ell$-adic valuation $|\cdot|_\ell$ is taken to be normalised by $|\ell|_\ell = \ell^{-1}$. If $\Gamma$ is a group acting on an Abelian group $A$, we denote by $A^\Gamma$ invariants and by $A_\Gamma$ coinvariants. By $X^{(i)}$, we denote the set of codimension-$i$ points of a scheme $X$, and by $|X|$ the set of closed points. For an Abelian variety $A$, we denote its Poincaré bundle by $\mathcal{P}_A$.

2 The L-function

The main theorem Theorem 2.44 of this section is a conditional result on the conjecture of Birch and Swinnerton-Dyer over higher dimensional bases over finite fields.

The results in this section are a generalisation of results of Schneider [Sch82a, p. 134–138 and Sch82b, p. 496–498].

Let $k = \mathbb{F}_q$ be a finite field with $q = p^n$ elements and let $\ell \neq p$ be a prime. For a variety $X/k$ denote by $\hat{X}$ its base change to a separable closure $\overline{k}$.

Denote by Frob$_q$ the arithmetic Frobenius, the inverse of the geometric Frobenius as defined in [KW01], p. 5, and by $\Gamma$ the absolute Galois group of the finite base field $k$.

Let $X/k$ be a smooth projective geometrically connected variety of dimension $d$, and let $A/X$ be an Abelian scheme.

2.1 Preliminaries on étale cohomology

Definition 2.1. A $\mathbb{Q}_\ell[\Gamma]$-module is said to be pure of weight $n$ if all eigenvalues $\alpha$ of the geometric Frobenius automorphism $\text{Frob}_q^{-1}$ are algebraic integers which have absolute value $q^{n/2}$ under all embeddings $\iota : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

We often use the yoga of weights (without further mentioning):

Theorem 2.2. Let $f : X \to Y$ be a smooth proper morphism of schemes of finite type over $\mathbb{F}_q$ and $\mathcal{F}$ a smooth sheaf pure of weight $n$. Then $R^if_!\mathcal{F}$ is a smooth sheaf pure of weight $n + i$ for any $i$.

Proof. Apply Poincaré duality to [Del80, p. 138, Théorème 1].

Definition 2.3. Let $V$ be a $\mathbb{Z}_\ell[\Gamma]$-module. Its $i$-th Tate twist $V(i)$ is defined as $V(i) = V \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(i)$ where $\mathbb{Z}_\ell(i) = \lim\limits_{\to} \mathbb{Z}_\ell[\mu_{p^n}]$ if $i \geq 0$ and $\mathbb{Z}_\ell(i) = \mathbb{Z}_\ell(-i)\infty$ if $i < 0$. 

Lemma 2.4. Let $V, W$ be $\mathbb{Q}[\Gamma]$-modules pure of weight $m$ and $n$, respectively.
(a) The tensor product $V \otimes_{\mathbb{Q}} W$ is a $\mathbb{Q}[\Gamma]$-module pure of weight $m + n$.
(b) $\text{Hom}_{\mathbb{Q}}(V, W)$ is a $\mathbb{Q}[\Gamma]$-module pure of weight $n - m$. In particular, $V^\vee$ is pure of weight $-m$.
(c) The $i$-th Tate twist $V(i)$ is pure of weight $m - 2i$.

Proof. This follows from [Del80], p. 154, (1.2.5). □

Lemma 2.5. If $V$ is a $\mathbb{Q}[\Gamma]$-module pure of weight $\neq 0$, $V_{\Gamma} = V^\Gamma = 0$. More generally, if $V$ and $W$ are $\mathbb{Q}[\Gamma]$-modules pure of weights $m \neq n$, every $\Gamma$-morphism $V \to W$ is zero.

Proof. See [Jan10], p. 4, Fact 2. □

Definition 2.6. For $\pi : \mathcal{A} \to X$ let
\[ P_1(\mathcal{A}/X, t) = \det(1 - t \text{Frob}_q^{-1} | H^1(\overline{X}, R^1\pi_*\mathcal{A})). \]
and define the relative $L$-function of an Abelian scheme $\mathcal{A}/X$ by
\[ L(\mathcal{A}/X, s) = \frac{P_1(\mathcal{A}/X, q^{-s})}{P_0(\mathcal{A}/X, q^{-s})}. \]

For our purposes, it is better to consider the following $L$-function:

Definition 2.7. Let
\[ L_i(\mathcal{A}/X, t) = \det(1 - t \text{Frob}_q^{-1} | H^i(\overline{X}, V_i\mathcal{A})). \]

Remark 2.8. This definition is motivated in Remark 4.26 below.

Definition 2.9. An isogeny of group schemes is a surjective group scheme homomorphism with finite kernel.

Lemma 2.10. Let $G, G'$ be commutative group schemes over a scheme $S$ which are smooth and of finite type over $S$ with connected fibres and $\dim G = \dim G'$ and let $f : G' \to G$ be a morphism of commutative group schemes over $S$. If $f$ is flat (étale, respectively) then $\ker(f)$ is a flat (étale, respectively) group scheme over $S$, $f$ is quasi-finite, surjective and defines an epimorphism in the category of flat (étale, respectively) sheaves over $S$.

Proof. (This is the (corrected) exercise 2.19 in [Mil80], p. 67, II §2.) Since $\ker(f) \to S$ is the base change of $f$ along the unit-section of $G$, it is flat (étale, respectively). That $f$ is surjective and quasi-finite can be checked fibrewise for $s \in S$. By the flatness and [BLR90], p. 178, §7.3 Lemma 1, we have that $f_s$ is finite and flat. So the image of $f_s$ is open and closed in $G_s$. Since $G_s$ is connected by assumption, $f_s$ must be surjective.

Now let $T$ be an $S$-scheme, $g \in \text{Hom}_S(T, G)$ and $T'$ the fibre product of $G'$ and $T$ along $f$ and $g$:

\[
\begin{array}{ccc}
T' & \xrightarrow{g'} & G' \\
\downarrow f' & & \downarrow f \\
T & \xrightarrow{g} & G
\end{array}
\]

Then the base change $f'$ of $f$ is again flat (étale, respectively) and surjective, and so is a covering in the stated topology. Hence, then the base change $g' \in \text{Hom}_S(T', G')$ of $g$ is a local lift of $g$ in that topology. So the claim follows. □

Lemma 2.11. Let $S$ be a scheme and $f : G \to S$ be a smooth commutative group scheme over $S$ and $n$ an integer invertible on $S$. Then the multiplication map $[n] : G \to G$ is étale and the $n$-torsion subgroup $G[n] := \ker([n]) \to S$ is an étale group scheme over $S$.

If, furthermore, $f$ is of finite type with connected fibres, then $[n]$ is surjective and induces an epimorphism in the category of étale sheaves over $S$.

Proof. For the first statement use [BLR90], p. 179, §7.3 Lemma 2 (b). Note that the assumption “of finite type” is not needed here (see also [SGA3] II 3.9.4). The morphism $\ker([n]) \to S$ is just the base change of $[n]$ along the unit-section. For the second part apply Lemma 2.10. □
Theorem 2.12. Let \( f : \mathcal{A} \to \mathcal{A}' \) be an \( X \)-isogeny of Abelian schemes. The Weil pairing
\[
\langle \cdot, \cdot \rangle_f : \ker(f) \times_X \ker(f') \to \mathbb{G}_m
\]
is a non-degenerate and biadditive pairing of finite flat \( X \)-group schemes, i.e. it defines a canonical \( X \)-isomorphism
\[
\ker(f') \xrightarrow{\sim} (\ker(f))^\ell.
\]
Moreover, it is functorial in \( f \).

If \( X = \text{Spec} \, k \), it induces a perfect pairing of torsion-free finitely generated \( \mathbb{Z}_\ell[\Gamma]\)-modules.

\[
T_\ell \mathcal{A} \times T_\ell (\mathcal{A}') \to \mathbb{Z}_\ell(1)
\]
(2.1)

\[
\text{Hom}(T_\ell \mathcal{A}, \mathbb{Z}_\ell) = T_\ell(\mathcal{A}')(-1)
\]
(2.2)

Proof. See [Mum70], p. 186 (for Abelian varieties) and [Oda69], p. 66 f., Theorem 1.1 (for Abelian schemes).

Theorem 2.13. Let \( K \) be an arbitrary field, \( \ell \neq \text{char} \, K \) be prime and \( A/K \) an Abelian variety. Let \( \bar{A} = A \times_K \bar{K} \).

Then we have an isomorphism of (\( \ell \)-adic discrete) \( G_K \)-modules, equivalently, by [Mil80], p. 53, Theorem II.1.9, of (\( \ell \)-adic) étale sheaves on \( \text{Spec} \, K \),

\[
T_\ell(A) = H^1(\bar{A}, \mathbb{Z}_\ell)^\vee.
\]

In particular, \( T_\ell(A) \) has weight \(-1\).

Proof. Consider the Kummer sequence

\[
1 \to \mu_{\ell^n} \to G_m \xrightarrow{\ell^n} G_m \to 1
\]
on \( \bar{A} \). Taking étale cohomology, one gets an exact sequence of \( G_K \)-modules

\[
0 \to G_m(\bar{A})/\ell^n \to H^1(\bar{A}, \mu_{\ell^n}) \to H^1(\bar{A}, G_m)[\ell^n] \to 0.
\]

Since \( \Gamma(\bar{A}, \mathcal{O}_A) = \bar{K} \) is separably closed and \( \ell \neq \text{char} \, K \), \( G_m(\bar{A}) \) is \( \ell \)-divisible (one can extract \( \ell \)-th roots), and hence

\[
H^1(\bar{A}, \mu_{\ell^n}) \xrightarrow{\sim} H^1(\bar{A}, G_m)[\ell^n] = \text{Pic}(\bar{A})[\ell^n] = \text{Pic}^0(\bar{A})[\ell^n],
\]

the latter equality since \( \text{NS}(\bar{A}) \) is torsion-free by [Mum70], p. 178, Corollary 2. Taking Tate modules \( \lim_n \) yields

\[
H^1(\bar{A}, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} T_\ell \text{Pic}^0(\bar{A}),
\]

so (the first equality coming from the perfect Weil pairing (2.2))

\[
\text{Hom}(T_\ell A, \mathbb{Z}_\ell(1)) = T_\ell(A^\vee) = H^1(\bar{A}, \mathbb{Z}_\ell(1)),
\]

so

\[
(T_\ell A)^\vee = \text{Hom}(T_\ell A, \mathbb{Z}_\ell) = H^1(\bar{A}, \mathbb{Z}_\ell),
\]

so

\[
T_\ell A = H^1(\bar{A}, \mathbb{Z}_\ell)^\vee.
\]

Alternatively, \( \pi_1(A, 0) = \prod \ell T_\ell(A) \) by [Mum70], p. 171, and \( H^1(\bar{A}, \mathbb{Z}_\ell) = \text{Hom}(\pi_1(A, 0), \mathbb{Z}_\ell) \) by [Kel16], p. 231, Proposition 4.14.

Remark 2.14. Note that both \( T_\ell(-) \) and \( H^1(-, \mathbb{Z}_\ell)^\vee \) are covariant functors.

Theorem 2.15. Let \( S \) be a locally Noetherian scheme, \( \pi : \mathcal{A} \to S \) be a projective Abelian scheme over \( S \). Let \( \ell \) be a prime number invertible on \( S \). Then we have a canonical isomorphism \( R^1 \pi_* \mathbb{Z}_\ell(1) = T_\ell \mathcal{A}^\vee \) as \( \ell \)-adic étale sheaves on \( S \). In particular, \( T_\ell \mathcal{A} \) has weight \(-1\).
Proof. Applying the functor $\pi_*$ on the exact Kummer sequence

$$1 \rightarrow \mu_{\ell^n} \rightarrow G_{m, sf} / \ell^n \rightarrow G_{m, sf} \rightarrow 1$$

of étale sheaves on $\mathcal{A}'$, we get an exact sequence

$$1 \rightarrow \pi_* G_{m, sf} / \ell^n \rightarrow R^1 \pi_* \mu_{\ell^n} \rightarrow R^1 \pi_* G_{m, sf}[\ell^n] \rightarrow 0.$$ 

of étale sheaves on $S$. The first term will vanish by following arguments. Since $\pi : \mathcal{A}' \rightarrow S$ is proper and its geometric fibres are integral by definition we get the isomorphism $\mathcal{O}_S = \pi_* \mathcal{O}_{\mathcal{A}'}$ by the Stein factorization (cf. [GW10], p. 348, Theorem 12.68). Hence we have $G_{m, S} = \pi_* G_{m, sf}$. But since $\ell$ is invertible on $S$ the map $\ell^n : G_{m, S} \rightarrow G_{m, S}$ is an epimorphism and we get

$$\pi_* G_{m, sf} / \ell^n = G_{m, S} / \ell^n = 1.$$ 

For the last term in the above sequence by [BLR90], p. 203, § 8.1 we get the canonical isomorphism $R^1 \pi_* G_{m, sf} = \text{Pic}_{\mathcal{A}'/S}$ since $\pi$ is smooth and proper. Note that, since $\mathcal{A}' \rightarrow S$ is projective and flat with integral fibres the Picard scheme exists by [FGI05], p. 263, Theorem 9.4.8. Let $\text{NS}_{\mathcal{A}'/S}$ be defined by the short exact sequence of étale sheaves:

$$0 \rightarrow \text{Pic}^0_{\mathcal{A}'/S} \rightarrow \text{Pic}_{\mathcal{A}'/S} \rightarrow \text{NS}_{\mathcal{A}'/S} \rightarrow 0.$$ 

Here $\text{Pic}^0_{\mathcal{A}'/S}$ is the identity component of $\text{Pic}_{\mathcal{A}'/S}$ and coincides with the dual Abelian scheme $\mathcal{A}'$ by [BLR90], p. 234, § 8.4 Theorem 5. This implies that $\text{Pic}_{\mathcal{A}'/S}$ is a smooth commutative group scheme over $S$. Taking $\ell^n$-torsion, which is left exact, we get a short exact sequence:

$$0 \rightarrow \text{Pic}^0_{\mathcal{A}'/S}[\ell^n] \rightarrow \text{Pic}_{\mathcal{A}'/S}[\ell^n] \rightarrow \text{NS}_{\mathcal{A}'/S}[\ell^n].$$

We now prove that $\text{Pic}^0_{\mathcal{A}'/S}[\ell^n] \rightarrow \text{Pic}_{\mathcal{A}'/S}[\ell^n]$ is an isomorphism by looking at the stalks. Since the first two groups are étale over $S$, by Lemma [2.11] it suffices to look at the sequence over the geometric points $s$ of $S$ by [Mil80], p. 34, Proposition I.4.1. But by [Mum70], p. 165, IV § 19 Theorem 3, Corollary 2, the group $\text{NS}_{\mathcal{A}'}(s)$ is a finitely generated free abelian group (since we are over a field) and its torsion part vanishes. So, all together, we have the isomorphisms:

$$R^1 \pi_* \mu_{\ell^n} = R^1 \pi_* G_{m, sf}[\ell^n] = \text{Pic}_{\mathcal{A}'/S}[\ell^n] = \text{Pic}^0_{\mathcal{A}'/S}[\ell^n] = \mathcal{A}'[\ell^n].$$

By taking the projective limit over all $n$ we then get the claim: $R^1 \pi_* \mathbb{Z}_\ell(1) = T_{\mathcal{A}'/S}$.

The statement on the weight follows from Lemma $2.14$ and Theorem $2.2$: $\mathbb{Z}_\ell(1)$ has weight $-2$ and $1 - 2 = -1$. 

Lemma 2.16. Let $f : A \rightarrow B$ be an isogeny of Abelian varieties over a field $k$ and $\ell \neq \text{char } k$. Then $f$ induces an Galois equivariant isomorphism $V_{i}A \sim \rightarrow V_{i}B$ of rationalised Tate modules.

Proof. There is the exact sequence of étale sheaves over $k$

$$0 \rightarrow \ker(f) \rightarrow A \rightarrow B \rightarrow 0.$$ 

This induces an exact sequence of $G_{\ell}$-modules

$$0 \rightarrow \ker(f)(k) \rightarrow A(k) \rightarrow B(k) \rightarrow 0.$$ 

Since for an abelian group $M$, one has $T_{\ell}M = \text{Hom}(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}, M)$, applying $\text{Hom}(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}, -)$ to the above exact sequence yields (writing, by abuse of notation, $T_{\ell}A$ for $T_{\ell}A(k)$)

$$0 \rightarrow T_{\ell} \ker(f)(k) \rightarrow T_{\ell}A \rightarrow T_{\ell}B \rightarrow \text{Ext}^1(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}, \ker(f)(k)).$$

Since $\ker(f)$ is a finite group scheme, we have $T_{\ell} \ker(f)(k) = 0$. Since $T_{\ell}A$ and $T_{\ell}B$ have the same rank as $f$ is an isogeny (or since $\text{Ext}^1(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}, \ker(f)(k))$ is finite), tensoring with $\mathbb{Q}_{\ell}$ yields the desired isomorphism. 

Proposition 2.17. Let $f : \mathcal{A}' \rightarrow \mathcal{B}$ be an isogeny of Abelian schemes over $S$ and $\ell$ invertible on $S$. Then $f$ induces an isomorphism $V_{i} \mathcal{A}' \sim \rightarrow V_{i} \mathcal{B}$. 

Proof. We check the isomorphism $V_t \mathcal{A} \to V_t \mathcal{B}$ on stalks. Let $\pi : \mathcal{A} \to S$ and $\pi' : \mathcal{B} \to S$ be the structure morphisms of the dual Abelian schemes and $\pi_*, \pi'_*$ the base changes of $\pi, \pi'$ by $\{x\} \to S$. By Theorem 2.15 we have $V_t \mathcal{A} = R^1\pi_*\mathcal{Q}_t(1)$ and $V_t \mathcal{B} = R^1\pi'_*\mathcal{Q}_t(1)$. Since $\pi$ and $\pi'$ are proper, by proper base change [Mil86a], p. 224, Corollary VI.2.5 $(\mathcal{Z}_t(1))$ is an inverse limit of the torsion sheaves $\mu_n)$. $(V_t \mathcal{A})_x = R^1\pi_*\mathcal{Q}_t(1)_x = R^1\pi_*\mathcal{Q}_t(1)$ and analogously for $\mathcal{B}$. So one can assume $S$ is the spectrum of a field. Then the statement is just Lemma 2.16.

Lemma 2.18. Let $\pi : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an $\ell$-adic sheaf on $X$. Then $R^i\pi_*(\mathcal{F}(n)) = (R^i\pi_*\mathcal{F})(n)$. 

Proof. We have 

$$R^i\pi_*(\mathcal{F}(n)) = R^i\pi_*(\mathcal{F} \otimes \mathcal{Z}_t(n))$$

$$= R^i\pi_*(\mathcal{F} \otimes \pi^*\mathcal{Z}_t(n)) \quad \text{since } \pi^*\mu_n = \mu_n$$

$$= R^i\pi_*(\mathcal{F}) \otimes \mathcal{Z}_t(n) \quad \text{by the projection formula [Mil80], p. 260, Lemma VI.8.8 since } \mathcal{Z}_t(n) \text{ is flat}$$

$$= (R^i\pi_*\mathcal{F})(n).$$

Proposition 2.19. Let $X$ be a normal Noetherian integral scheme and $\mathcal{A}/X$ an Abelian scheme. Then there is an isogeny $\mathcal{A} \to \mathcal{A}'$ (a polarisation).

Proof. Since an isogeny is defined fibrewise, we have to show that there exists a relatively ample line bundle for $\mathcal{A}/X$ since ample line bundles induce polarisations (see [Mil86a], p. 126, §13). This follows from [Ray70], p. 170, Théorème XI.1.13 and by property (A) in [Ray70], p. 159, Definition XI.1.2 and by the existence of an ample line bundle on the generic fibre [Mil86a], p. 114, Corollary 7.2.

Remark 2.20. Note that 

$$P_r(\mathcal{A}/X, q^{-s}) = \det(1 - q^{-s} \text{Frob}_q^{-1} | H^i(\tilde{X}, R^1\pi_*\mathcal{Q}_t))$$

$$= \det(1 - q^{-s} \text{Frob}_q^{-1} | H^i(\tilde{X}, V_t(\mathcal{A}'(1)(-1)))) \quad \text{by Theorem 2.15}$$

$$= \det(1 - q^{-s} \text{Frob}_q^{-1} | H^i(\tilde{X}, V_t(\mathcal{A}'(1)(-1))) \quad \text{by Lemma 2.18}$$

$$= \det(1 - q^{-s} \text{Frob}_q^{-1} | H^i(\tilde{X}, V_t(\mathcal{A}'(1))))$$

$$= \det(1 - q^{-s} \text{Frob}_q^{-1} | H^i(\tilde{X}, V_t(\mathcal{A}))) \quad \text{by Proposition 2.17 and Proposition 2.19}$$

$$= L_i(\mathcal{A}/X, q^{-s+1}),$$

so the vanishing order of $P_r(\mathcal{A}/X, q^{-s})$ at $s = 1$ is equal to the vanishing order of $L_i(\mathcal{A}/X, t)$ at $t = q^{-1+1} = q^0 = 1$, and the respective leading coefficients agree.

The following is a generalisation of [Sch82a], p. 134–138 and [Sch82b], p. 496–498.

For the definition of an $\ell$-adic sheaf see [FKSS], p. 122, Definition 12.6. For $\ell$ invertible on $X$, $T_\ell\mathcal{A} = (\mathcal{A}[\ell^n])_{n \in \mathbb{N}}$ is an $\ell$-adic sheaf.

The following lemma is well known.

Lemma 2.21. Isogenous Abelian varieties over a finite field $k$ have the same number of points.

Theorem 2.22. Let $X$ be a proper scheme over a separably closed or finite field $K$ and $\mathcal{F}$ be a constructible $\ell$-adic sheaf on $X$. Then $H^q(X, \mathcal{F})$ is finite for all $q \geq 0$.

Note that [Mil86a], p. 224, Corollary VI.2.8 is wrong in general (consider $X = \text{Spec} \mathbb{Q}$ with $H^1(\text{Spec} \mathbb{Q}, \mu_n) = \mathbb{Q}^\times/n!)$.

Proof. By the proper base change theorem [Mil86a], p. 223, Theorem VI.2.1 the claim follows for separably closed fields. For a finite field $K$ with absolute Galois group $\Gamma$, the claim follows by passing to a separable closure $\overline{K}$ of $K$ and the use of Hochschild-Serre spectral sequence $H^p(\Gamma, H^q(X, \mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$ with $X := X \times_K \overline{K}$, which degenerates by [Wei76], p. 124, Exercise 5.2.1 because of $\text{cd}(\Gamma) = 1$ by [NSW00], p. 69, (1.6.13) (ii) as $\Gamma = \mathbb{Z}$ into short exact sequences 

$$0 \to H^{r-1}(\tilde{X}, \mathcal{F})_\Gamma \to H^r(X, \mathcal{F}) \to H^r(\tilde{X}, \mathcal{F})^\Gamma \to 0$$

with the outer groups being finite by the case of a separably closed ground field.
Lemma 2.23. Let $X$ be a variety over a finite field $k$ with absolute Galois group $\Gamma$. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$, $\mathcal{F} = \varprojlim_n \mathcal{F}_n$ be an $\ell$-adic sheaf. For every $i$, there is a short exact sequence

$$0 \to H^{-1}(\bar{X}, \mathcal{F})_\Gamma \to H^i(X, \mathcal{F}) \to H^i(\bar{X}, \mathcal{F})_\Gamma \to 0 \quad (2.4)$$

with $H^i(\bar{X}, \mathcal{F})$ and $H^i(X, \mathcal{F})$ finitely generated $\mathbb{Z}_\ell$-modules.

The following argument is a generalisation of [Mil88], p. 78, Lemma 3.4.

**Proof.** Since $\Gamma = \hat{\mathbb{Z}}$ has cohomological dimension 1 by [NSW00], p. 69, (1.6.13) (ii), we get from the Hochschild-Serre spectral sequence for $\bar{X}/X$ [Mil80], p. 106, Remark III.2.21 (b) by [Wei97], p. 124, Exercise 5.2.1 short exact sequences for every $n$ and $i$

$$0 \to H^{-1}(\bar{X}, \mathcal{F}_n)_\Gamma \to H^i(X, \mathcal{F}_n) \to H^i(\bar{X}, \mathcal{F}_n)_\Gamma \to 0.$$

Since all involved groups are finite (because the two outer groups are finite by Theorem 2.22 since $\bar{X}/k$ is proper over $k$ separably closed and $\mathcal{F}_n$ is constructible by definition of an $\ell$-adic sheaf), the system satisfies the Mittag-Leffler condition, so taking the projective limit yields an exact sequence

$$0 \to \varprojlim_n(H^{-1}(\bar{X}, \mathcal{F}_n)_\Gamma) \to H^i(X, \mathcal{F}) \to \varprojlim_n(H^i(\bar{X}, \mathcal{F}_n)_\Gamma) \to 0.$$ 

Write $M[n]$ for $H^i(\bar{X}, \mathcal{F}_n)$. Breaking the exact sequence

$$0 \to M[n]_\Gamma \to M[n] \xrightarrow{\text{Frob}^{-1}} M[n]_\Gamma \to 0$$

into two short exact sequences and applying $\varprojlim_n$, one obtains, setting $Q[n] = (\text{Frob}^{-1})M[n]$, two exact sequences

$$0 \to \varprojlim_n(M[n]^\Gamma) \to \varprojlim_n(M[n]) \xrightarrow{\text{Frob}^{-1}} \varprojlim_n(Q[n]) \to \varprojlim_n(M[n]^\Gamma) \quad (2.5)$$

$$0 \to \varprojlim_n(Q[n]) \to \varprojlim_n(M[n]) \to \varprojlim_n(M[n]_\Gamma) \to \varprojlim_n(Q[n]) \quad (2.6)$$

Since the $M[n]$, and hence the $M[n]^\Gamma$ are finite (argument as above), they form a Mittag-Leffler system, and hence one gets from (2.5) an exact sequence

$$0 \to \varprojlim_n(M[n]^\Gamma) \to \varprojlim_n(M[n]) \xrightarrow{\text{Frob}^{-1}} \varprojlim_n(Q[n]) \to 0.$$ 

Similarly, the $Q[n] \subseteq M[n]$ are finite, and hence

$$0 \to \varprojlim_n(Q[n]) \to \varprojlim_n(M[n]) \to \varprojlim_n(M[n]_\Gamma) \to 0$$

is exact from (2.6). Combining the above two short exact sequences, one gets the exactness of

$$0 \to \varprojlim_n(M[n]^\Gamma) \to \varprojlim_n(M[n]) \xrightarrow{\text{Frob}^{-1}} \varprojlim_n(M[n]_\Gamma) \to \varprojlim_n(M[n]_\Gamma) \to 0,$$

which shows that for all $i$

$$\varprojlim_n(H^i(\bar{X}, \mathcal{F}_n)^\Gamma) = \varprojlim_n(M[n]^\Gamma) = \ker(\varprojlim_n(M[n]) \xrightarrow{\text{Frob}^{-1}} \varprojlim_n(M[n])) = H^i(\bar{X}, \mathcal{F})^\Gamma$$

$$\varprojlim_n(H^i(\bar{X}, \mathcal{F}_n)_\Gamma) = \varprojlim_n(M[n]_\Gamma) = \coker(\varprojlim_n(M[n]) \xrightarrow{\text{Frob}^{-1}} \varprojlim_n(M[n])) = H^i(\bar{X}, \mathcal{F})_\Gamma,$$

which is what we wanted. \qed
This implies
\[
H^{2d}(\bar{X}, T_{\ell}\mathcal{O}) \xrightarrow{\sim} H^{2d+1}(X, T_{\ell}\mathcal{O})
\]
(2.7)
since \(H^{2d+1}(\bar{X}, T_{\ell}\mathcal{O}) = 0\) by [Mil80], p. 221, Theorem VI.1.1 as \(\dim \bar{X} = d\). Because of \(H^i(\bar{X}, T_{\ell}\mathcal{O}) = 0\) for \(i > 2d\) for the same reason, it follows that \(H^i(X, T_{\ell}\mathcal{O}) = 0\) for \(i > 2d + 1\). Furthermore, one has
\[
Z_i = (Z_i)_\Gamma = H^{2d}(\bar{X}, Z_i(d)) \xrightarrow{\sim} H^{2d+1}(X, Z_i(d)),
\]
(2.8)
the second equality by Poincaré duality [Mil80], p. 276, Theorem VI.1.1 (a) and the isomorphism by Lemma 2.23 since \(H^{2d+1}(\bar{X}, Z_i(d)) = 0\) by [Mil80], p. 221, Theorem VI.1.1 as \(\dim \bar{X} = d\).

**Definition 2.24.** Let \(f : A \to B\) be a homomorphism of Abelian groups. If \(\ker(f)\) and \(\coker(f)\) are finite, \(f\) is called a quasi-isomorphism, in which case we define
\[
q(f) = \frac{|\ker(f)|}{|\coker(f)|}.
\]

The following lemma is crucial for relating special values of \(L\)-functions and orders of cohomology groups.

**Lemma 2.25.** Let \(\text{Frob}\) be a topological generator of \(\Gamma\) and \(M\) be a finitely generated \(\mathbb{Z}_\ell\)-module with continuous \(\Gamma\)-action. Then the following are equivalent:
1. \(\det(1 - \text{Frob} | M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \neq 0\).
2. \(H^0(\Gamma, M) = M^\Gamma\) is finite.
3. \(H^1(\Gamma, M)\) is finite.

If one of these holds, we have \(H^1(\Gamma, M) = M_\Gamma\) and
\[
|\det(1 - \text{Frob} | M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)| = \frac{|(\text{Frob})^M|}{|M^\Gamma|} = \frac{|\ker(1 - \text{Frob})|}{|\text{coker}(1 - \text{Frob})|} = q(1 - \text{Frob})^{-1}.
\]

**Corollary 2.26.** Let \(\text{Frob}\) be a topological generator of \(\Gamma\) and \(M\) be a finitely generated \(\mathbb{Z}_\ell\)-module with continuous \(\Gamma\)-action. If \(M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\) has weight \(\neq 0\), \(M^\Gamma\) and \(M_\Gamma\) are finite.

**Proof.** Since \(M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\) has weight \(\neq 0\), \(\text{Frob}\) has all eigenvalues \(\neq 1 = q^{0/2}\), hence \(\det(1 - \text{Frob} | M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)\) is \(\neq 0\), so the corollary follows from Lemma 2.25. \(\square\)

### 2.2 The cohomological formula

**Corollary 2.27.** If \(i \neq 1\), \(H^i(\bar{X}, T_{\ell}\mathcal{O})^\Gamma\) and \(H^i(\bar{X}, T_{\ell}\mathcal{O})_\Gamma\) are finite, and one has
\[
|L_i(\mathcal{O}/X, 1)| \ell = \frac{|H^i(\bar{X}, T_{\ell}\mathcal{O})^\Gamma|}{|H^i(\bar{X}, T_{\ell}\mathcal{O})_\Gamma|}.
\]

**Proof.** This follows from Lemma 2.25 and Theorem 2.13 which is \(\neq 0\) if \(i \neq 1\). \(\square\)

After Corollary 2.27 one can concentrate on \(i = 1\).

**Lemma 2.28.** Infinite groups in the short exact sequences in (2.4) can only occur in the following two sequences:
\[
0 \longrightarrow H^1(\bar{X}, T_{\ell}\mathcal{O})^\Gamma \xrightarrow{\beta} H^2(X, T_{\ell}\mathcal{O}) \longrightarrow H^2(\bar{X}, T_{\ell}\mathcal{O})^\Gamma \longrightarrow 0
\]
(2.9)
\[
0 \longrightarrow H^0(\bar{X}, T_{\ell}\mathcal{O})^\Gamma \longrightarrow H^1(X, T_{\ell}\mathcal{O}) \xrightarrow{\alpha} H^1(\bar{X}, T_{\ell}\mathcal{O})^\Gamma \longrightarrow 0
\]

Here, \(f\) is induced by the identity on \(H^1(\bar{X}, T_{\ell}\mathcal{O})\). The morphisms \(\alpha\) and \(\beta\) are quasi-isomorphisms, i.e. they have finite kernel and cokernel, and \(\alpha\) is surjective and \(\beta\) is injective.
Proof. Since $T_i\mathcal{A}$ has weight $-1$ by Theorem 2.15, $H^i(\bar{X}, T_i\mathcal{A})$ has weight $i-1$ by Theorem 2.2. So the conditions of Lemma 2.25 are fulfilled for the $\Gamma$-module $M = H^i(\bar{X}, T_i\mathcal{A})$ and $i \neq 1$. Therefore infinite groups in the short exact sequences in (2.4) can only occur in the two sequences of diagram (2.9). Since $H^0(\bar{X}, T_i\mathcal{A})_\Gamma$ and $H^0(\bar{X}, T_i\mathcal{A})_\Gamma$ are finite (having weight $2-1 \neq 0$ and $0-1 \neq 0$, so Corollary 2.26 applies), $\alpha$ and $\beta$ are quasi-isomorphisms, i.e. they have finite kernel and cokernel, and $\alpha$ is surjective and $\beta$ is injective.

Recall from Definition 2.7 that

$$L_1(\mathcal{A}/X, t) = \det(1 - t \operatorname{Frob}_q^{-1} | H^1(\bar{X}, V_\mathcal{A})).$$

Define $\tilde{L}_1(\mathcal{A}/X, t)$ and $\rho$ by

$$\rho = \operatorname{ord} L_1(\mathcal{A}/X, t) \in \mathbb{N},$$

$$L_1(\mathcal{A}/X, t) = (t - 1)^\rho \cdot \tilde{L}_1(\mathcal{A}/X, t).$$

Note that $\tilde{L}_1(\mathcal{A}/X, 1) \neq 0$ and $\tilde{L}_1(\mathcal{A}/X, t) \in \mathbb{Q}[t]$.

The idea is that for infinite cohomology groups $H^1(\bar{X}, T_i\mathcal{A})$, one should insert a regulator term $q(f)$ or $q((\beta f\alpha)_{n-tors})$ with $(\beta f\alpha)_{n-tors}$ induced by $\beta f\alpha$ by modding out torsion.

**Lemma 2.29.** One has $\rho = \operatorname{rk}_{\mathbb{Z}_p} H^1(X, T_i\mathcal{A})$ with the $\rho$ from (2.10) iff in (2.9) is a quasi-isomorphism. In this case,

$$|\tilde{L}_1(\mathcal{A}/X, 1)|^{-1} = q(f) = \frac{|\operatorname{coker} f|}{|\operatorname{ker} f|}$$

and

$$|\tilde{L}_1(\mathcal{A}/X, 1)|^{-1} = q((\beta f\alpha)_{n-tors}) \cdot \frac{|H^0(\bar{X}, T_i\mathcal{A})|}{|H^2(X, T_i\mathcal{A})|} \cdot \frac{|H^2(X, T_i\mathcal{A})_\Gamma|}{|H^1(X, T_i\mathcal{A})_\Gamma|}$$

with $\tilde{L}_1(\mathcal{A}/X, t)$ from (2.11).

**Proof.** By writing $\operatorname{Frob}_q^{-1}$ in Jordan normal form, one sees that $\rho$ is equal to

$$\dim_{\mathbb{Q}} \bigcup_{n \geq 1} \ker(1 - \operatorname{Frob}_q)^n \geq \dim_{\mathbb{Q}} \ker(1 - \operatorname{Frob}_q^{-1}) = \dim_{\mathbb{Q}} H^1(\bar{X}, V_\mathcal{A})^\Gamma,$$

i.e.

$$\rho \geq \dim_{\mathbb{Q}} H^1(\bar{X}, V_\mathcal{A})^\Gamma,$$

and that equality holds iff the operation of the Frobenius on $H^1(\bar{X}, V_\mathcal{A})$ is semi-simple at 1, i.e.

$$\dim_{\mathbb{Q}} \bigcup_{n \geq 1} \ker(1 - \operatorname{Frob}_q^{-1}) \leq \dim_{\mathbb{Q}} \ker(1 - \operatorname{Frob}_q^{-1}),$$

i.e. the generalised eigenspace at 1 equals the eigenspace, which is equivalent to $f_{\mathbb{Q}}$, in (2.9) being an isomorphism, i.e. $f$ being a quasi-isomorphism.

From (2.9), since $H^0(\bar{X}, T_i\mathcal{A})_\Gamma$ is finite, one sees that

$$\dim_{\mathbb{Q}} H^1(\bar{X}, V_\mathcal{A})^\Gamma = \operatorname{rk}_{\mathbb{Z}_p} H^1(\bar{X}, T_i\mathcal{A}) = \operatorname{rk}_{\mathbb{Z}_p} H^1(X, T_i\mathcal{A}).$$

Hence, the first statement follows.

Assuming $f$ being a quasi-isomorphism, one has by Lemma 2.25 and arguing as in [Sch82a, p. 136, proof of Lemma 3]

$$|\tilde{L}_1(\mathcal{A}/X, 1)| = \frac{|(\operatorname{Frob}_q - 1)H^1(\bar{X}, T_i\mathcal{A})|}{|(\operatorname{Frob}_q - 1)^2H^1(\bar{X}, T_i\mathcal{A})|}$$

$$= \frac{|(\operatorname{Frob}_q - 1)H^1(\bar{X}, T_i\mathcal{A})|}{|\operatorname{coker} f|} = q(f)^{-1}.$$
For the second equation,
\[ q(f) = \frac{q((\beta f)_{\text{tors}})}{q(\beta)} \cdot q((\beta f)_{\text{tors}}) \text{ by } [\text{Latt66}], \text{ p. 306-19–306-20, Lemma z.1–z.4} \]
\[ = \frac{1}{|H^2(X, T_\ell A)|^2} \cdot \left| \frac{[H^2(X, T_\ell A)]_{\text{tors}}}{[H^1(X, T_\ell A)]_{\text{tors}}} \right| \cdot q((\beta f)_{\text{tors}}) \]
\[ = q((\beta f\alpha)_{\text{tors}}) \cdot \left| \frac{[H^0(X, T_\ell A)]}{[H^2(X, T_\ell A)]^2} \right| \cdot \left| \frac{[H^2(X, T_\ell A)]_{\text{tors}}}{[H^1(X, T_\ell A)]_{\text{tors}}} \right| \text{ since coker}(\alpha) = 0. \]

**Lemma 2.30.** Let \( \ell \neq p \) be invertible on \( X \). Then there is an exact sequence
\[ 0 \to A(X) \otimes Q/\mathbb{Z}_\ell \to H^1(X, A'[\ell^\infty]) \to H^1(X, A)[\ell^\infty] \to 0. \]

**Proof.** Since \( \ell \) is invertible on \( X \), one has the short exact Kummer sequence of étale sheaves \( 0 \to A'[\ell^m] \to A'[\ell^m] \to 0 \), which induces
\[ 0 \to A(X)/\ell^m \to H^1(X, A'[\ell^m]) \to H^1(X, A)[\ell^m] \to 0. \] (2.12)

Passing to the colimit \( \varinjlim_n \) yields the result.

**Remark 2.31.** This reminds us of the exact sequence
\[ 0 \to A(K)/n \to \text{Sel}(n)(A/K) \to \text{III}(A/K)[n] \to 0 \]
for an Abelian variety \( A \) over a global field \( K \).

**Lemma 2.32.** Let \( A \) be an Abelian \( \ell \)-primary torsion group such that \( A[\ell] \) is finite. Then \( A \) is a cofinitely generated \( Z_\ell \)-module.

**Proof.** Equip \( A \) with the discrete topology. Applying Pontryagin duality to \( 0 \to A[\ell] \to A \rightarrow A \) gives us that \( A^D/\ell \) is finite, hence by [NSW00], p. 179, Proposition 3.9.1 \( (A^D \text{ being profinite as a dual of a discrete torsion group}) \), \( A^D \) is a finitely generated \( Z_\ell \)-module, hence \( A \) a cofinitely generated \( Z_\ell \)-module.

Recall the definition of the Tate-Shafarevich group, \( \text{III}(A[X]) = H^1_{\text{et}}(X, A[X]) \).

**Lemma 2.33.** Let \( \ell \) be invertible on \( X \). Then the \( Z_\ell \)-corank of \( \text{III}(A[X])[\ell^\infty] \) is finite.

**Proof.** From (2.12), one sees that \( H^1(X, A'[\ell])[\ell] \) is finite as it is a quotient of \( H^1(X, A'[\ell]) \) and \( A'[\ell]/X \) is constructible, and the cohomology of a constructible sheaf on a proper variety over a finite field is finite by Theorem 2.22. Hence \( \text{III}(A[X])[\ell^\infty] \) is cofinitely generated by Lemma 2.32.

For an \( \ell \)-adic sheaf \( \mathcal{F} \), denote by \( \mathcal{F}(n) \) the \( n \)-th Tate twist of \( \mathcal{F} \), \( \mathcal{F}(n) := \mathcal{F} \otimes_{Z_\ell} Z_\ell(n) \), see Definition 2.3

**Lemma 2.34.** Let \( X/k \) be proper over \( k \) separably closed or finite, and let \( \ell \) be invertible on \( X \). There is a long exact sequence
\[ \ldots \to H^i(X, T_\ell A(n)) \to H^i(X, T_\ell A(n)) \otimes_{Z_\ell} Q/\ell \to H^i(X, A'[\ell^\infty](n)) \to \ldots \]
which induces isomorphisms
\[ H^{i-1}(X, A'[\ell^\infty](n))_{\text{n-div}} \sim H^i(X, T_\ell A(n))_{\text{tors}} \]
and short exact sequences
\[ 0 \to H^i(X, T_\ell A(n))_{\text{n-tors}} \to H^i(X, T_\ell A(n)) \otimes_{Z_\ell} Q/\ell \to H^i(X, A'[\ell^\infty](n))_{\text{div}} \to 0. \]

**Proof.** Consider for \( m, m' \in \mathbb{N} \) invertible on \( X \) the short exact sequence of étale sheaves
\[ 0 \to A'[n](n) \to A'[mn'](n) \xrightarrow{m} A'[n'](n) \to 0. \]
Setting \( m = \ell^m, m' = \ell^{m'} \), the associated long exact sequence is
\[ \ldots \to H^i(X, A'[\ell^m](n)) \to H^i(X, A'[\ell^{m+1}](n)) \to H^i(X, A'[\ell^m](n)) \to \ldots \]
Passing to the projective limit $\varprojlim_{\nu} \otimes_{\mathbb{Q}_\nu}$ and then to the inductive limit $\varinjlim_{\nu}$ yields the desired long exact sequence since all involved cohomology groups are finite by Theorem 2.22 since $X/k$ is proper over a separably closed or finite field and our sheaves are constructible. Here, we use that $\varinjlim$ is exact on finite groups, see [Wei97], p. 83, Proposition 3.5.7 and Exercise 3.5.2.

For the second statement, consider the exact sequence
\[
H^{-1}(X, T_{i}\mathcal{A}(n)) \otimes \mathbb{Z}_{\ell} \xrightarrow{g} H^{-1}(X, \mathcal{A}[\ell^\infty](n)) \xrightarrow{d} H^{0}(X, T_{i}\mathcal{A}(n)) \xrightarrow{\delta} H^{1}(X, T_{i}\mathcal{A}(n)) \otimes \mathbb{Z}_{\ell}.
\]
Since $H^{i}(X, T_{i}\mathcal{A}(n))$ is a finitely generated $\mathbb{Z}_{\ell}$-module (since $(\mathcal{A}[\ell^r])_{n\in\mathbb{N}}$ is an $\ell$-adic sheaf) and $g$ is induced by the identity, we have $ker g = H^{i}(X, T_{i}\mathcal{A}(n))_{\text{tors}}$. Note that $H^{i}(X, T_{i}\mathcal{A}(n)) \cong \mathbb{Z}_{\ell}^{rk} \oplus H^{i}(X, T_{i}\mathcal{A}(n))_{\text{tors}}$ and that the codomain of $g$ is isomorphic to $\mathbb{Q}_{\ell}^{rk}$. Since $H^{-1}(X, T_{i}\mathcal{A}(n))$ is a finitely generated $\mathbb{Z}_{\ell}$-module and $H^{-1}(X, \mathcal{A}[\ell^\infty](n))$ is a cofinitely generated $\ell$-torsion module isomorphic to $(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{rk} \oplus H^{0}(X, T_{i}\mathcal{A}(n))_{\text{tors}}$, so the divisible part is $(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{rk}$, we have $im f = H^{1}(X, \mathcal{A}[\ell^\infty](n))_{\text{div}}$. The claim follows from the exactness of the sequence.

For the third statement, consider the exact sequence
\[
H^{i}(X, T_{i}\mathcal{A}(n)) \xrightarrow{\delta} H^{i}(X, T_{i}\mathcal{A}(n)) \otimes \mathbb{Z}_{\ell} \xrightarrow{g} H^{i}(X, \mathcal{A}[\ell^\infty](n)).
\]
Since $H^{i}(X, T_{i}\mathcal{A}(n))$ is a finitely generated $\mathbb{Z}_{\ell}$-module (since $(\mathcal{A}[\ell^r])_{n\in\mathbb{N}}$ is an $\ell$-adic sheaf) and $g$ is induced by the identity, we have $ker g = H^{i}(X, T_{i}\mathcal{A}(n))_{\text{tors}}$. Note that $H^{i}(X, T_{i}\mathcal{A}(n)) \cong \mathbb{Z}_{\ell}^{rk} \oplus H^{i}(X, T_{i}\mathcal{A}(n))_{\text{tors}}$ and that the codomain of $g$ is isomorphic to $\mathbb{Q}_{\ell}^{rk}$. Since $H^{i}(X, T_{i}\mathcal{A}(n))$ is a finitely generated $\mathbb{Z}_{\ell}$-module and $H^{i}(X, \mathcal{A}[\ell^\infty](n))$ is a cofinitely generated $\ell$-torsion module isomorphic to $(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{rk} \oplus H^{i}(X, T_{i}\mathcal{A}(n))_{\text{tors}}$, so the divisible part is $(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{rk}$, we have $im f = H^{i}(X, \mathcal{A}[\ell^\infty](n))_{\text{div}}$. The claim follows from the exactness of the sequence.

**Theorem 2.35** (Mordell-Weil). Let $K$ be a field finitely generated over its prime field and $A/K$ an Abelian variety. Then the Mordell-Weil group $A(K)$ is a finitely generated Abelian group.

**Proof.** See [Cona], p. 3, Theorem 2.1.

Note that $\mathcal{A}(X) = A(K)$ by the Néron mapping property:

**Theorem 2.36** (The Néron model). Let $S$ be a regular, Noetherian, integral, separated scheme with $g : \{\eta\} \hookrightarrow S$ the inclusion of the generic point. Let $\mathcal{A}/S$ be an Abelian scheme. Then
\[
\mathcal{A} \xrightarrow{\sim} g_{*}g^{\ast}\mathcal{A}
\]
as sheaves on $S_{\text{sm}}$.

**Proof.** See [Kel16], p. 222, Theorem 3.3.

**Lemma 2.37.** Assume $\ell$ is invertible on $X$. Then one has the following identities for the étale cohomology groups of $X$:

\[
\text{H}^{i}(X, T_{i}\mathcal{A}) = 0 \quad \text{for} \quad i \neq 1, 2, \ldots, 2d + 1
\]
\[
\text{H}^{1}(X, T_{i}\mathcal{A})_{\text{tors}} = \text{H}^{0}(X, \mathcal{A}[\ell^\infty])_{\text{n-div}} = \text{H}^{0}(X, \mathcal{A})[\ell^{\infty}]
\]
\[
\text{H}^{2}(X, T_{i}\mathcal{A})_{\text{tors}} = \text{H}^{1}(X, \mathcal{A}[\ell^\infty])_{\text{n-div}}
\]
\[
\text{H}^{1}(X, \mathcal{A}[\ell^\infty])_{\text{n-div}} = \text{III}(\mathcal{A}/X)[\ell^{\infty}] \quad \text{if III}(\mathcal{A}/X)[\ell^{\infty}] \text{ is finite}
\]

**Proof.** (2.13): For $i > 2d + 1$ this follows from (2.4) (using the fact that $\text{H}^{i}(X, T_{i}\mathcal{A}) = 0$ for $i > 2d$, as noted earlier below (2.7)), and it holds for $i = 0$ since $\text{H}^{0}(X, \mathcal{A}[\ell^r]) \subset \mathcal{A}(X)_{\text{tors}}$ is finite (since $\mathcal{A}(X)$ is a finitely generated Abelian group by the Mordell-Weil theorem Theorem 2.35 and the Néron mapping property Theorem 2.36) hence its Tate-module is trivial.

(2.14) and (2.15): From Lemma 2.34 we get
\[
\text{H}^{1}(X, T_{i}\mathcal{A})_{\text{tors}} = \text{H}^{i-1}(X, \mathcal{A}[\ell^\infty])_{\text{n-div}}
\]
The desired equalities follow by plugging in $i = 1, 2$.

Further, one has $\text{H}^{0}(X, \mathcal{A}[\ell^r])_{\text{n-div}} = \text{H}^{0}(X, \mathcal{A})[\ell^\infty]_{\text{tors}}$ in (2.14) because $\text{H}^{0}(X, \mathcal{A}[\ell^\infty])$ is cofinitely generated by the Mordell-Weil theorem and the Néron mapping property Theorem 2.36.

Finally, (2.16) holds since by Lemma 2.30 $\text{H}^{1}(X, \mathcal{A}[\ell^r])_{\text{n-div}} = \text{H}^{1}(X, \mathcal{A})[\ell^{\infty}]$ if the latter is finite, and this equals $\text{III}(\mathcal{A}/X)[\ell^{\infty}]$. 

\[\square\]
Now we have two pairings given by cup product in cohomology
\[
\langle \cdot, \cdot \rangle_f : H^1(X, T_t \mathcal{O}_f)_{\text{n-tors}} \times H^{d-1}(X, T_t(\mathcal{O}^f)(d-1))_{\text{n-tors}} \to H^{d}(X, Z_t(d)) \to H^{d}(\bar{X}, Z_t(d)) = Z_t, \quad (2.17)
\]
\[
\langle \cdot, \cdot \rangle_t : H^2(X, T_t \mathcal{O}_f)_{\text{n-tors}} \times H^{2d-1}(X, T_t(\mathcal{O}^f)(d-1))_{\text{n-tors}} \to H^{2d+1}(X, Z_t(d)) = Z_t. \quad (2.18)
\]

Lemma 2.38. Let $A, A'$ and $B$ finitely generated free $\mathbb{Z}_t$-modules. Consider the commutative diagram
\[
\begin{array}{ccc}
A & \times & B \\
\downarrow f & & \downarrow \langle \cdot, \cdot \rangle \\
A' & \times & B
\end{array}
\]
where $\langle \cdot, \cdot \rangle$ is a non-degenerate pairing.

Then $f$ is a quasi-isomorphism iff $\langle \cdot, \cdot \rangle$ is non-degenerate, and in this case one has
\[
q(f) = \frac{\left| \det \langle \cdot, \cdot \rangle \right|^{-1}}{\left| \det \langle \cdot, \cdot \rangle \right|_\ell}.
\]

Proof. Since the $\mathbb{Z}_t$-modules are finitely generated free, the pairings are non-degenerate if they are perfect after tensoring with $\mathbb{Q}_t$. So $f$ is a quasi-isomorphism iff $f \otimes_{\mathbb{Z}_t} \mathbb{Q}_t$ is an isomorphism iff $f_{\mathbb{Q}_t} : \text{Hom}(A_{\mathbb{Q}_t}, \mathbb{Q}_t) \to \text{Hom}(A_{\mathbb{Q}_t}, \mathbb{Q}_t) = B_{\mathbb{Q}_t}$ (the latter equality coming from $\langle \cdot, \cdot \rangle_{\mathbb{Q}_t}$ being perfect, a non-degenerate pairing of finite dimensional vector spaces is perfect, and being non-degenerate is preserved by localisation) is an isomorphism iff $\langle \cdot, \cdot \rangle_{\mathbb{Q}_t}$ is perfect.

The statement on $q(f)$ follows by considering the dual diagram
\[
\begin{array}{ccc}
A^\vee & \longrightarrow & B^\vee \\
\downarrow f & & \downarrow \langle \cdot, \cdot \rangle \\
A'^{\vee} & \longrightarrow & B'^{\vee}
\end{array}
\]
from [Tat66], p. 433f., Lemma 2.1 and Lemma 2.2. \hfill \square

Lemma 2.39. Recall the maps $\alpha, \beta, f$ from diagram (2.9). The pairing $\langle \cdot, \cdot \rangle_\ell$ is non-degenerate. The regulator term $q((\beta f \alpha)_{\text{n-tors}})$ is defined iff $f$ is a quasi-isomorphism, and then equals
\[
\frac{\left| \det \langle \cdot, \cdot \rangle_\ell \right|^{-1}}{\left| \det \langle \cdot, \cdot \rangle_\ell \right|_\ell}
\]
where both pairings are non-degenerate. Conversely, if the pairing $\langle \cdot, \cdot \rangle_\ell$ is non-degenerate, $f$ is a quasi-isomorphism.

Proof. Using $H^{2d+1}(X, Z_t(d)) = Z_t$ and $H^{2d}(\bar{X}, Z_t(d)) = Z_t$ by (2.8), there is a commutative diagram of pairings
\[
\begin{array}{ccc}
H^2(X, T_t \mathcal{O}_f)_{\text{n-tors}} & \times & H^{2d-1}(X, T_t(\mathcal{O}^f)(d-1))_{\text{n-tors}} \\
\downarrow \beta_{\text{n-tors}} & \cong & \downarrow \langle \cdot, \cdot \rangle_\ell \\
(H^1(\bar{X}, T_t \mathcal{O}_f)^!)_{\text{n-tors}} & \times & (H^{2d-1}(\bar{X}, T_t(\mathcal{O}^f)(d-1))^!)_{\text{n-tors}} \\
\downarrow f_{\text{n-tors}} & & \downarrow \cong \leftarrow Z_t \\
(H^1(\bar{X}, T_t \mathcal{O}_f)^!)_{\text{n-tors}} & \times & (H^{2d-1}(\bar{X}, T_t(\mathcal{O}^f)(d-1))^!)_{\text{n-tors}} \\
\downarrow \alpha_{\text{n-tors}} & \cong & \downarrow \cong \\
H^1(X, T_t \mathcal{O}_f)_{\text{n-tors}} & \times & H^{2d-1}(X, T_t(\mathcal{O}^f)(d-1))_{\text{n-tors}} \\
\downarrow \langle \cdot, \cdot \rangle_\ell & & \downarrow \langle \cdot, \cdot \rangle_\ell \\
Z_t & & Z_t
\end{array}
\]
where the maps $\alpha_{n\text{-tors}}, \beta_{n\text{-tors}}$ and $f_{n\text{-tors}}$ are induced by the maps $\alpha$ resp. $\beta$ resp. $f$ in diagram (2.9). Note that by Lemma 2.28 $\alpha_{n\text{-tors}}$ is an isomorphism and $\beta_{n\text{-tors}}$ is injective with finite cokernel.

As in [Sch22a, p. 137, (5)], this diagram is commutative with the pairing in the second line non-degenerate. (By Poincaré duality [MHS0], p. 276, Theorem VI.11.1 (using that $T_\ell Z$ is a smooth sheaf since the $Z[\ell^n]$ are étale), the pairing in the second line is not degenerate, hence the pairing in the first line is too since $\beta$ is a quasi-isomorphism. The upper right and the lower right arrows [note that these are the same morphism] are isomorphisms since their kernel is by Lemma 2.23 $H^{2d-4}(X, T_\ell Z[d-1])_{n\text{-tors}}$, which equals 0 for weight reasons by Corollary 2.26 $(2d-2) + (1-1) - (d-1) = -1 \neq 0$; the lower left arrow $\alpha_{n\text{-tors}}$ is an isomorphism since $\alpha$ is a quasi-isomorphism since it is surjective by (2.9) with finite kernel again by (2.9) (the kernel $H^0(X, T_\ell Z)$ is finite since $H^0(X, T_\ell Z)$ has weight $-1 + 0 \neq 0$ by Theorem 2.15 so the $\Gamma$-invariants are finite by Corollary 2.26).

Hence if $f$ is a quasi-isomorphism, the pairing $\langle \cdot, \cdot \rangle_\ell$ is non-degenerate by Lemma 2.38 and then the claimed equality for the regulator $g((\beta f)_{n\text{-tors}}) = |\text{coker}(\beta f)_{n\text{-tors}}|$ follows since $\text{ker}(\beta f)_{n\text{-tors}} = 0$ by Lemma 2.38.

Conversely, if $\langle \cdot, \cdot \rangle_\ell$ is non-degenerate, $f$ is a quasi-isomorphism by Lemma 2.38.

Lemma 2.40. Let $\ell$ be invertible on $X$. Then one has a short exact sequence

$$0 \to \mathcal{A}(X) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}_{\ell} \xrightarrow{\delta} H^1(X, T_\ell \mathcal{A}) \to \lim_n \left( H^1(X, \mathcal{A}[\ell^n]) \right) \to 0.$$ 

If $\text{III}((\mathcal{A}/X)[\ell^\infty]) = H^1(X, \mathcal{A})[\ell^\infty]$ is finite, $\delta$ induces an isomorphism

$$\mathcal{A}(X) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}_{\ell} \xrightarrow{\sim} H^1(X, T_\ell \mathcal{A}).$$

Proof. Since $\ell$ is invertible on $X$, the short exact Kummer sequence of étale sheaves

$$0 \to \mathcal{A} \otimes \mathbb{Z}_{\ell} \xrightarrow{\mathcal{A}[\ell^n]} \mathcal{A} \to 0$$

induces a short exact sequence

$$0 \to \mathcal{A}(X)/\ell^n \xrightarrow{\delta} H^1(X, \mathcal{A}[\ell^n]) \to H^1(X, \mathcal{A})[\ell^n] \to 0$$

in cohomology, and passing to the limit $\lim_n$ gives us the desired short exact sequence since the $\mathcal{A}(X)/\ell^n$ are finite by the Mordell-Weil theorem Theorem 2.33 and the Néron mapping property Theorem 2.36 so they satisfy the Mittag-Leffler condition and $\lim_n \mathcal{A}(X)/\ell^n = 0$.

The second claim follows from the short exact sequence and since the Tate module of a finite group is trivial.

Lemma 2.41. One has $\rho \geq \text{rk}_{\mathbb{Z}_{\ell}} H^1(X, T_\ell \mathcal{A})$. Consider the following statements:

1. $\langle \cdot, \cdot \rangle_\ell$ is non-degenerate.
2. $f$ is a quasi-isomorphism.
3. Equality holds $\rho = \text{rk}_{\mathbb{Z}_{\ell}} H^1(X, T_\ell \mathcal{A})$.
4. The canonical injection $\mathcal{A}(X) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}_{\ell} \xrightarrow{\delta} H^1(X, T_\ell \mathcal{A})$ is surjective.
5. The $\ell$-primary part of the Tate-Shafarevich group $\text{III}(\mathcal{A}/X)[\ell^\infty]$ is finite.

Then $1 \iff 2 \iff 3$ and $4 \iff 5$; further $3 \iff 4$ assuming $\rho = \text{rk}_{\mathbb{Z}_{\ell}} \mathcal{A}(X)$.

Furthermore, the following are equivalent:

(a) $\rho = \text{rk}_{\mathbb{Z}_{\ell}} \mathcal{A}(X)$
(b) $\langle \cdot, \cdot \rangle_\ell$ is non-degenerate and the $\ell$-primary part of the Tate-Shafarevich group $\text{III}(\mathcal{A}/X)[\ell^\infty]$ is finite.

Proof. 1 $\iff$ 2: See Lemma 2.39.

2 $\iff$ 3: This is Lemma 2.29.

3 $\iff$ 4: One has $\rho = \text{rk}_{\mathbb{Z}_{\ell}} \mathcal{A}(X) = \text{rk}_{\mathbb{Z}_{\ell}} (\mathcal{A}(X) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}_{\ell})$ and by Lemma 2.40 $\text{rk}_{\mathbb{Z}_{\ell}} (\mathcal{A}(X) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}_{\ell}) \leq \text{rk}_{\mathbb{Z}_{\ell}} H^1(X, T_\ell \mathcal{A})$, so this is an equality iff $\delta$ in 4. is onto.

4 $\iff$ 5: One has that $\text{lim}_{\ell} H^1(X, \mathcal{A}[\ell^n]) = T_\ell (H^1(X, \mathcal{A}))$ is trivial iff $H^1(X, \mathcal{A})[\ell^\infty] = \text{III}(\mathcal{A}/X)[\ell^\infty]$ is finite since $\text{III}(\mathcal{A}/X)[\ell^\infty]$ is a cofinitely generated $\mathbb{Z}_{\ell}$-module by Lemma 2.33.

(a) $\implies$ (b): Since $\delta$ in (4) is injective, one has $\text{rk}_{\mathbb{Z}_{\ell}} \mathcal{A}(X) \leq \text{rk}_{\mathbb{Z}_{\ell}} H^1(X, T_\ell \mathcal{A}) \leq \rho$. Therefore, $\rho = \text{rk}_{\mathbb{Z}_{\ell}} \mathcal{A}(X)$ implies equality, and (3) and (4) follow, so (1)–(5) hold. (b) $\implies$ (a): from (b) follows (5) $\implies$ (4) and (1) $\implies$ (2) $\implies$ (3), so from (4) one gets $\mathcal{A}(X) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}_{\ell} \xrightarrow{\sim} H^1(X, T_\ell \mathcal{A})$, but by (3), $\rho = \text{rk}_{\mathbb{Z}_{\ell}} H^1(X, T_\ell \mathcal{A}) = \text{rk}_{\mathbb{Z}_{\ell}} \mathcal{A}(X)$. 


Remark 2.42. We have
\[ 1 - q^{1-s} = 1 - \exp(-(s-1) \log q) = (\log q)(s-1) + O((s-1)^2) \quad \text{for } s \to 1 \]
using the Taylor expansion of exp.

Define \( c \) by
\[
L(\mathcal{A}/X, s) \sim c \cdot (1 - q^{1-s})^\rho \\
\sim c \cdot (\log q)^\rho (s-1)^\rho \quad \text{for } s \to 1,
\]
(2.19)
see Remark 2.42. Note that \( c \in \mathbb{Q} \) since \( L(\mathcal{A}/X, s) \) is a rational function with \( \mathbb{Q} \)-coefficients in \( q^{-s} \), and \( c \neq 0 \) since \( \rho \) is the vanishing order of the \( L \)-function at \( s = 1 \) by definition of \( \rho \) and the Riemann hypothesis.

Corollary 2.43. If \( \rho = \mathrm{rk}_\mathbb{Z}_\ell \mathbb{H}^1(X, T_\ell \mathcal{A}) \), then
\[
|\mathcal{C}|^{-1} = q((\beta f \alpha)_{\text{tors}}) \cdot \frac{\left| \mathbb{H}^2(X, T_\ell \mathcal{A})_{\text{tors}} \right|}{\left| \mathbb{H}^1(X, T_\ell \mathcal{A}) \right|}.
\]
Proof. Using Lemma 2.29 for \( L_1(\mathcal{A}/X, t) \) and Corollary 2.27 for \( L_0(\mathcal{A}/X, t) \), one gets
\[
|\mathcal{C}|^{-1} = q((\beta f \alpha)_{\text{tors}}) \cdot \frac{\left| \mathbb{H}^0(X, T_\ell \mathcal{A}) \right|}{\left| \mathbb{H}^2(X, T_\ell \mathcal{A}) \right|}.
\]
\[
= q((\beta f \alpha)_{\text{tors}}) \cdot \frac{\left| \mathbb{H}^1(X, T_\ell \mathcal{A}) \right|}{\left| \mathbb{H}^2(X, T_\ell \mathcal{A}) \right|}.
\]
For \( 0 = \mathbb{H}^0(X, T_\ell \mathcal{A}) \sim H^0(X, T_\ell \mathcal{A}) \cdot \mathbb{Q} \) use (2.4) with \( i = 0 \) and (2.13).

Theorem 2.44 (The conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher-dimensional bases, cohomological version). Recall that \( \rho = \text{ord}_{s=1} L(\mathcal{A}/X, t) \) is the analytic rank of \( \mathcal{A}/X \). One has \( \rho \geq \text{rk}_\mathbb{Z}_\ell \mathbb{H}^1(X, T_\ell \mathcal{A}) \geq \text{rk}_\mathbb{Z} \mathcal{A}(K) \).

(a) For some \( \beta \neq \text{char } k \), \( \langle \cdot, \cdot \rangle_1 \) is non-degenerate and \( \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \) is finite. If these hold, we have for all \( \ell \neq p \)
\[
|\mathcal{C}|^{-1} = \frac{\left| \det(\langle \cdot, \cdot \rangle_1) \right|}{\left| \mathbb{H}^2(\mathcal{A}/X)[\ell^\infty] \right|} \cdot \frac{\left| \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \right|}{\left| \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \right|}.
\]
where \( c \) is defined by (2.19), and the prime-to-p torsion \( \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \cdot \text{non-p} \) is finite.

Proof. Note that \( \rho = \text{ord}_{s=1} L_1(\mathcal{A}/X, t) = \text{ord}_{s=1} L_1(\mathcal{A}/X, t) \) by Remark 2.8, and that \( \text{rk}_\mathbb{Z}_\ell \mathbb{H}^1(X, T_\ell \mathcal{A}) \geq \text{rk}_\mathbb{Z} \mathcal{A}(X) \) by the injection from Lemma 2.41(4). The first statement is Lemma 2.41(a) \( \iff \) (b). Now identify the terms in Corollary 2.43 using Lemma 2.37 (cohomology groups) and Lemma 2.39 (regulator).

By Theorem 2.44(b) for \( \ell \implies \) (a) independent of \( \ell \implies \) (b) for \( \ell \neq p \), \( \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \) is finite for every \( \ell \neq p \). But since \( c \neq 0 \), and by the relation of \( |c|^{-1} \) and \( \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \), the prime-to-p torsion is finite.

Remark 2.45. (a) For example, the conjecture is true if \( L(\mathcal{A}/X, 1) \neq 0 \) since one then has \( 0 = \rho \geq \text{rk}_\mathbb{Z} \mathcal{A}(X) \geq 0 \).

(b) The (determinants of the) pairings \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_\ell \) are identified below: One has \( \det(\langle \cdot, \cdot \rangle_1) = 1 \) and \( \det(\langle \cdot, \cdot \rangle_\ell) \) is the regulator, see especially Remark 3.31.

(c) For the vanishing of \( H^2(X, T_\ell \mathcal{A}) \) see Remark 4.32 below.

(d) Assume \( \mathbb{H}(\mathcal{A}/X)[\ell^\infty] \) finite. Then, the pairing \( \langle \cdot, \cdot \rangle_\ell \) has determinant 1, see the discussion in subsection 3.2 below, and is thus non-degenerate. The pairing \( \langle \cdot, \cdot \rangle_\ell \) equals the height pairing, see subsection 3.1 below, and is therefore non-degenerate by [Cona], p. 35, Theorem 9.15.
Remark 2.46. This remark is about the independence of the analytic rank \( \text{ord}_s = 1 L(\mathcal{A}/X, s) \) on the model \( \mathcal{A}/X \) of \( A/K \).

Note that the vanishing order of \( L(\mathcal{A}/X, s) \) at \( s = 1 \), the analytic rank, only depends on \( L_1(\mathcal{A}/X, s) \) since \( L_0(\mathcal{A}/X, 1) = \det(1 - \text{Frob}_q^{-1} | \mathcal{H}^0(\mathcal{X}, \mathcal{V}_L)) \neq 0 \) by Lemma 2.15 below, its invariants \( \mathcal{H}_1^0(\mathcal{X}, \mathcal{V}_L) \) are finite by Corollary 2.26. Furthermore, the vanishing order of \( L_1 \) at \( s = 1 \) only depends on the generic fibre \( A/K \) (and not on the model \( X \)) assuming the conjecture of Birch and Swinnerton-Dyer for \( \mathcal{A}/X \) by an a posteriori argument: If the conjecture holds, by Theorem 2.44 the vanishing order at \( s = 1 \) of \( L(\mathcal{A}/X, s) \) equals the (algebraic) rank of \( A(K) \).

For \( \dim X = 1 \), there is a canonical model \( X \) of \( K \). In contrast, for higher dimensional \( X \), there is no canonical model (one can e.g. blow up smooth centres), and the special \( L \)-value depends on the model. If every birational morphism of smooth projective \( k \)-varieties of dimension \( d \) is given by a sequence of monoidal transformations (e.g. for surfaces, see Har83, p. 412, Theorem V.5.5 over algebraically closed fields), the vanishing order of \( L_1(\mathcal{A}/X, 1) = \det(1 - \text{Frob}_q^{-1} | \mathcal{H}^1(\mathcal{X}, \mathcal{V}_L)) \) is independent of the model of \( X \) by calculation of the étale cohomology of blow-ups of torsion sheaves [Sta8 section 0EW3]: If \( X' \) is the blow-up of \( X \) in a closed point \( \tilde{Z} \) with exceptional divisor \( \tilde{E} \cong \mathbb{P}^{d-1} \), then there is an exact sequence of proper varieties over \( k \)

\[
\mathcal{H}^0(\mathcal{E}, \mathcal{V}_L) \to \mathcal{H}^1(\mathcal{X}, \mathcal{V}_L) \to \mathcal{H}^1(\mathcal{X}, \mathcal{V}_L) \oplus \mathcal{H}^1(\mathcal{Z}, \mathcal{V}_L) \to \mathcal{H}^1(\mathcal{E}, \mathcal{V}_L).
\]

Here, \( \mathcal{H}^1(\mathcal{E}, \mathcal{V}_L) = 0 \) since \( c_d \mathcal{Z} = 0 \), \( \mathcal{H}^1(\mathcal{E}, \mathcal{V}_L) \) is pure of weight \( 0 - 1 \neq 0 \) and \( \mathcal{A}|^\mathcal{E} \) is finite étale, so \( \mathcal{H}^1(\mathcal{E}, \mathcal{V}_L) = \mathcal{H}^1(\mathcal{P}^d, \mathcal{O}(1)) = 0 \).

3 Cohomological and height pairings

3.1 The pairing \( \langle \cdot, \cdot \rangle_{\ell} \)

The goal of this subsection is to prove:

Theorem 3.1. The cohomological pairing \( \langle \cdot, \cdot \rangle_{\ell} \) from Theorem 2.44 equals the generalised Bloch pairing and the canonical Néron-Tate height pairing (for their definition see below) up to the integral hard Lefschetz defect (see Definition 3.6).

For the definition of a generalised global field see Con, p. 27, Definition 8.1. Let us recall Conrad’s height pairing for generalised global fields from Con, p. 27 ff. Let \( X/k \) be a smooth projective geometrically connected variety. Let \( k = \mathbb{F}_q \) be the finite ground field and \( K = k(X) \) the function field of \( X \). Choose a closed immersion \( \iota : X \hookrightarrow \mathbb{P}^N_k \). For \( x \in X^{(1)} \) let

\[
eq q^{-\deg_\iota, \iota^{-1}(x)}.
\]

For the definition of the degree of a closed subscheme of projective space see Har83, p. 52. Then the discrete valuations

\[
\| \cdot \|_{x, \iota} : c_{x, \iota}^{\text{ord}, \iota} : K \to \mathbb{Z},
\]

where \( x \in X^{(1)} \), satisfy the product formula by Har83, p. 146, Exercise II.6.2 (d). Conrad calls the system of these valuations the structure of a generalised global field on \( K \). This induces a height function

\[
h_{K, \iota, \iota} : \mathbb{P}_K^N \to \mathbb{R}, h_{K, \iota, \iota}(t_0 : \ldots : t_n) = \frac{1}{[K' : K]} \sum_{i=0}^{\max(\log \| t_i \|_{x, \iota})} \geq 0
\]

on projective spaces over \( K \), see Con, p. 29 ff, with the finitely many lifts \( v' \) of \( v = x \in X^{(1)} \) to \( K'/K \) finite, where \( K \subseteq K' \subseteq K \) is a finite subextension over \( K \) that contains the \( t_i \) and we canonically endow \( K' \) with a structure of generalised global field via the algebraic method as in Con, p. 28; this is independent of the choice of \( K' \), see Con, p. 30.

Now, we construct a canonical height pairing

\[
\hat{h}_x : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathbb{R}
\]

as follows. If \( \mathcal{L} \) is a very ample line bundle on a projective \( K \)-variety \( X \), the induced closed \( K \)-immersion

\[
i_{\mathcal{L}} : X \hookrightarrow \mathbb{P}(\mathcal{H}^0(X, \mathcal{L}))
\]
defines a height function

\[ h_{K, \mathcal{L}, t} := h_{K, \mathcal{P}(X, \mathcal{L}), t} \circ \iota_{\mathcal{L}} : X(\overline{K}) \to \mathbb{R}, \]

where \( h_{K, \mathcal{P}(X, \mathcal{L}), t} = h_{K, n, t} \) if \( \dim_K H^0(X, \mathcal{L}) = n \). By linearity, since one can write any line bundle on \( X \) as a difference of two very ample line bundles (see [HS00], p. 186, l. 8), this extends as in [HS00], p. 184, Theorem B.3.2 (Weil’s height machine) to a homomorphism

\[ \text{Pic}(X) \to \mathbb{R}^{X(\overline{K})}/O(1), \]

where \( O(1) := \{ f : X(\overline{K}) \to \mathbb{R} : f \text{ is bounded} \} \subset \mathbb{R}^{X(\overline{K})} \) is the subvector space of bounded functions.

Now let \( \mathcal{A}/X \) be an Abelian scheme. In this case, one can, by the Tate limit argument, define a canonical height pairing, taking values in \( \mathbb{R}^{X(\overline{K})} \), not modulo bounded functions,

\[ \hat{h}_{K, \mathcal{L}} : \mathcal{A}(X) = A(K) \to \mathbb{R} \]

or

\[ \hat{h}_{K, \mathcal{L}} : \mathcal{A}(X) \times \mathcal{A}(X) = A(K) \times A'(K) \to \mathbb{R}, \]

respectively as in [BG06], p. 284 ff. One has \( \hat{h}_{K, \mathcal{L}}(x, \mathcal{L}) = \hat{h}_{K, \mathcal{L}}(x) \).

**Proposition 3.2.** Let \( K \) be a generalised global field, \( A/K \) an Abelian variety and \( \mathcal{P} \in \text{Pic}(A \times A') \) the Poincaré bundle. Then

\[ \hat{h}_{\mathcal{P}}(x) = \hat{h}_{\mathcal{P}}(x, \mathcal{L}) \]

for \( x \in A(K) \) and \( \mathcal{L} \in A'(K) \).

**Proof.** See [BG06], p. 292, Corollary 9.3.7. \( \square \)

**Lemma 3.3.** Let \( \mathcal{A}/X \) be a projective Abelian scheme over a locally Noetherian scheme \( X \). Let \( x \in \mathcal{A}(X) \) and \( \mathcal{L} \in \text{Pic}^0(X) = \mathcal{A}(X) \). By the universal property of the Poincaré bundle [FGI 03], p. 262 f., Exercise 9.4.3, there is a unique \( X \)-morphism \( h : X \to \mathcal{A} \) such that \( \mathcal{L} = (\text{id}_{\mathcal{A}} \times h)^* \mathcal{P}_{\mathcal{A}}. \) Then \( x^* \mathcal{L} = (x, h)^* \mathcal{P}_{\mathcal{A}}. \)

**Proof.** Note that the map \( (x, h) : X \to \mathcal{A} \times_X \mathcal{A} \) factors as

\[ X \xrightarrow{x} \mathcal{A} \xrightarrow{(\text{id}_{\mathcal{A}} \times h)} \mathcal{A} \times_X \mathcal{A}. \]

Consequently, \( x^* \mathcal{L} = x^*(\text{id}_{\mathcal{A}} \times h)^* \mathcal{P}_{\mathcal{A}} = (x, h)^* \mathcal{P}_{\mathcal{A}}. \) \( \square \)

Let \( \ell \neq \text{char } k \) be prime. Now note that from Lemma 2.40, under the assumption \( \mathcal{I}(\mathcal{A}/X)[\ell^\infty] \) finite, one has an isomorphism

\[ \delta : \mathcal{A}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} H^1(X, T_{\ell, \mathcal{A}}). \]  

(3.1)

induced by the boundary map of the long exact sequence induced by the short exact Kummer sequence

\[ 0 \to \mathcal{A}[\ell^n] \to \mathcal{A} \to \mathcal{A} \to 0. \]

**Theorem 3.4** (hard Lefschetz for finite ground fields). Let \( k \) be a finite field, \( X \) a smooth projective scheme of pure dimension \( d \), \( \eta \in H^2(X, \mathbb{Z}_\ell(1)) \) the first Chern class of \( \mathcal{O}_X(1) \in \text{Pic}(X) \) (the image of \( \mathcal{O}_X(1) \) under the homomorphism \( \text{Pic}(X) \to H^2(X, \mathbb{Z}_\ell(1)) \)); this map comes from the Kummer sequence, see [Mil80], Proposition VI.10.1) and \( \mathcal{A}/X \) an Abelian scheme. Then the iterated cup products

\[ (\cup \eta)^j : H^{d-j}(X, V_{\ell, \mathcal{A}}) \to H^{d+j}(X, V_{\ell, \mathcal{A}}(i)) \]

are isomorphisms.

**Proof.** This follows from the hard Lefschetz theorem [BBDS2], p. 144, Théorème 5.4.10 for the projective morphism \( f : X \to \text{Spec } k \) since \( \mathcal{F} := V_{\ell, \mathcal{A}}[d] \) is a pure perverse sheaf: It is pure of weight \(-1\) by Theorem 2.15 and it is perverse since \( V_{\ell, \mathcal{A}} = R^1\pi_* \mathcal{Q}_x \) for the smooth projective morphism \( \pi : \mathcal{A} \to X \) is a smooth sheaf by proper and smooth base change [Mil80], p. 223, Corollary VI.2.2 and p. 230, Corollary VI.4.2, and if \( \mathcal{F} \) is a smooth sheaf on a smooth pure \( d \)-dimensional variety, then \( \mathcal{F}[d] \) is perverse by [KW01], p. 149, Corollary III.5.5. \( \square \)
Theorem 3.5 (integral hard Lefschetz for finite ground fields). The integral hard Lefschetz morphism

$$(\cup \eta)^{d-1} : H^i(X, T_\ell(A))(d-1)_{n\text{-tors}} \to H^{2d-1}(X, T_\ell(A)(d-1))_{n\text{-tors}}$$

is injective with finite cokernel.

Proof. By the hard Lefschetz theorem Theorem 3.4, it follows that the kernel and the cokernel tensored with $\mathbb{Q}_\ell$ are trivial, hence torsion, hence finite. Now note that all groups are taken modulo their torsion subgroup, so the kernel is trivial. $\square$

Definition 3.6. We call the order of the cokernel of the integral hard Lefschetz morphism from Theorem 3.5 the integral hard Lefschetz defect.

Here is an example where the integral hard Lefschetz morphism is an isomorphism (for varieties over an algebraically closed field):

Theorem 3.7. Let $A/k$ be an Abelian variety of dimension $d$ over a algebraically closed field with principal polarisation $\mathcal{L} \in \text{Pic}(A)$. Denote by $\vartheta \in H^2(A, \mathbb{Z}_\ell(1))$ the image of $\mathcal{L}$ under the homomorphism $\text{Pic}(A) \to H^2(A, \mathbb{Z}_\ell(1))$. Then the integral hard Lefschetz morphism $(\cup \vartheta)^{d-1} : H^1(A, \mathbb{Z}_\ell) \to H^{2d-1}(A, \mathbb{Z}_\ell(d-1))$ is an isomorphism.

We merely sketch the proof as this is not needed in the following.

Proof. Using that $\vartheta$ is a principal polarisation, write $\vartheta = \sum_{i=1}^d e_i \wedge e'_i$ in a symplectic basis (with respect to the Weil pairing $\wedge : T_\ell A \times T_\ell(A^\vee) \to \mathbb{Z}_\ell(1)$; using the principal polarisation, the Weil pairing becomes a symplectic pairing $T_\ell A \times T_\ell A \to \mathbb{Z}_\ell(1)$ by [Mil86a, p. 132, Lemma 16.2(e)] and use that the cohomology ring $H^*(A, \mathbb{Z}_\ell) = \wedge^* H^1(A, \mathbb{Z}_\ell)$ is an exterior algebra.

By [Mil86a, p. 130, one has $H^*(A, \mathbb{Z}_\ell) = (\wedge^* T_\ell A)^\vee$ (here we use that the ground field is algebraically closed).

Note that, via the identifications of the cohomology ring with the exterior algebra, proving that $(\cup \vartheta)^{d-1}$ is an isomorphism is equivalent to showing that this morphism sends a basis of $\wedge^1 T_\ell A$ to a basis of $\wedge^{2d-1} T_\ell A$. A basis of $\wedge^1 T_\ell A$ is $e_1, e'_1, \ldots, e_d, e'_d$, and a basis of $\wedge^{2d-1} T_\ell A$ is (a hat denotes the omission of a term)

$$e_1 \wedge e'_1 \wedge \ldots \wedge e_i \wedge e'_i \wedge \ldots \wedge e_d \wedge e'_d$$

and the same for $e'_i$ instead of $e_i$. Now,

$$\vartheta^{d-1} = \sum_{i=1}^d (e_1 \wedge e'_1 \wedge \ldots \wedge e_i \wedge e'_i \wedge \ldots \wedge e_d \wedge e'_d).$$

Thus,

$$e'_1 \wedge \vartheta^{d-1} = e_1 \wedge e'_1 \wedge \ldots \wedge e_i \wedge e'_i \wedge \ldots \wedge e_d \wedge e'_d$$

and the same for $e_i$, gives a basis of $\wedge^{2d-1} T_\ell A$. $\square$

Corollary 3.8. Let $\mathcal{A} = B \times_k X$ be a constant Abelian scheme with $X = A$ a principally polarised Abelian variety over a algebraically closed field. Then the integral hard Lefschetz morphism $(\cup \vartheta)^{d-1} : H^1(\mathcal{A}, \mathbb{Z}_\ell) \to H^{2d-1}(\mathcal{A}, \mathbb{Z}_\ell(d-1))$ is an integral isomorphism.

Proof. Note that $H^1(A, T_\ell \mathcal{A}) = H^1(A, \mathbb{Z}_\ell) \times T_\ell B$ by Lemma 4.12 and the projection formula. $\square$
Theorem 3.9. Let \( \ell \) be invertible on \( X \) and let \( \mathcal{A}/X \) be finite. Then there is a commutative diagram

\[
H^1(X, \mathcal{T}_X) \otimes_{\mathbb{Z}} H^{d-1}(X, T_r(\mathcal{O}^\ell)(d-1)) \otimes_{\mathbb{Z}} Z_t \xrightarrow{(\text{id}, \eta^d-1)} H^d(X, Z_t(d)) \xrightarrow{\text{pr}_1^*} H^2d(X, Z_t(d)) \xrightarrow{\text{pr}_2^*} H^2d(X, Z_t(d)) \xrightarrow{\sim} Z_t
\]

Here, \( \eta \in H^2(X, \mathcal{Z}_t(1)) \) is the cycle associated to \( \mathcal{O}_X(1) \in \text{Pic}(X) = CH^1(X) \) (\( X \) is regular) by \( \text{Pic}(X) \to H^2(X, \mathcal{Z}_t(1)) \) (this map comes from the Kummer sequence, see [Mil80], p. 271, Proposition VI.10.1), where \( \mathcal{O}_X(1) = \mathcal{O}_X^\ell(1) \) for the closed immersion \( i : X \hookrightarrow P_X^r \) which defines the structure of a generalised global field on the function field \( K = k(X) \) of \( X \). Further, \( \mathcal{O}(X) \otimes_{\mathbb{Z}} \mathcal{O}(X) \otimes_{\mathbb{Z}} Z_t = \mathcal{O}(X) \otimes_{\mathbb{Z}} \mathcal{O}(X) \otimes_{\mathbb{Z}} Z_t \to H^1(X, \mathcal{G}_m) \otimes_{\mathbb{Z}} Z_t \to H^1(X, \mathcal{G}_m) \otimes_{\mathbb{Z}} Z_t \) is the Yoneda Ext-pairing (the equality \( \mathcal{O}(X) = \text{Ext}^1_X(\mathcal{O}, \mathcal{G}_m) \) comes from the Barsotti-Weil formula Theorem 3.11). The pairing in the lower row is the generalised Néron-Tate canonical height pairing divided by \( -\log q \). The left vertical isomorphism \( (\delta, \delta^t) \) comes from [5.1], and the injection \( \cup \eta^d-1 \) from Theorem 3.3.

Commutativity of (1).

Lemma 3.10. The diagram

\[
H^1(X, \mathcal{T}_X) \otimes_{\mathbb{Z}} H^{d-1}(X, T_r(\mathcal{O}^\ell)(d-1)) \otimes_{\mathbb{Z}} Z_t \xrightarrow{(\text{id}, \eta^d-1)} H^d(X, Z_t(d)) \xrightarrow{\text{pr}_1^*} H^d(X, Z_t(d)) \xrightarrow{\text{pr}_2^*} H^d(X, Z_t(d)) \xrightarrow{\sim} Z_t
\]

commutes.

Proof. Let \( (x, y) \in H^1(X, \mathcal{T}_X) \otimes_{\mathbb{Z}} H^{d-1}(X, T_r(\mathcal{O}^\ell)(d-1)) \otimes_{\mathbb{Z}} Z_t \). Then \( \text{pr}_1^*((x \cup (y \cup \eta^d-1))) = \text{pr}_1^*((x \cup y) \cup \eta^d-1) \) for all \( x \in H^1(X, \mathcal{T}_X) \otimes_{\mathbb{Z}} Z_t \) and \( y \in H^1(X, \mathcal{T}_X) \otimes_{\mathbb{Z}} Z_t \) by the associativity of the cup product.

Commutativity of (2). Here, \( \delta : \mathcal{O}(X) \otimes_{\mathbb{Z}} Z_t \xrightarrow{\sim} H^1(X, T_r\mathcal{O}^\ell) \) is the morphism [3.1]; denote the analogous maps \( \delta^t : \mathcal{O}(X) \otimes_{\mathbb{Z}} Z_t \xrightarrow{\sim} H^1(X, T_r\mathcal{O}^\ell) \).

Theorem 3.11 (Barsotti-Weil formula). Let \( X \) be a locally Noetherian normal scheme. Then there is an isomorphism of fpff sheaves on \( X \)

\[
\delta \text{xt}^1_X(\mathcal{O}, \mathcal{G}_m) \xrightarrow{\sim} \delta^1.
\]

It is given on sections by mapping \( e : (1 \to \mathcal{G}_m) \to G \to \mathcal{A} \to 0 \) \( \in \text{Ext}^1(\mathcal{O}, \mathcal{G}_m) \) to \( G/\mathcal{A} \) considered as a \( \mathcal{G}_m \)-torsor, i.e. as a line bundle on \( \mathcal{A} \).

Proof. See [Mil86a], p. 121, 1 – 11 or [Oor66], p. III.18 – 1, Theorem III.18.1.

In the following, if we deal with Ext-groups or \( \delta \text{xt}\)-sheaves, we always mean them with respect to the fpff topology in order to have the Barsotti-Weil formula Theorem 3.11. Although we are also dealing with étale cohomology, there is no problem since by [Mil80], p. 116, Remark 3.11 (b) the étale and fpff cohomology of sheaves represented by smooth group schemes (we are using \( \mathcal{A}, \mathcal{A}[\ell^n], \mathcal{G}_m \) and \( \mu_{\ell^n} \) with \( \ell \) invertible on \( X \)) agree.

Note that one has an Yoneda Ext-pairing

\[
V : \text{Ext}^1(A, B) \times \text{Ext}^2(B, C) \to \text{Ext}^{1+2}(A, C),
\]
see [Mil80], p. 167; we will use this several times below. This induces pairings
\[ \nu : H^n(X, \mathcal{F}) \times \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G}) \to H^{n+i}(X, \mathcal{G}). \]
See also [GM03], p. 166f.

**Proposition 3.12.** Let \( \mathcal{A}/X \) be a projective Abelian scheme over a locally Noetherian scheme \( X \). Then the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{A}(X) & \times & \mathcal{A}^t(X) \\
\downarrow \cong & & \downarrow \cong \\
H^0(X, \mathcal{A}) & \times & \operatorname{Ext}^1_X(\mathcal{A}, G_m) \nu \to H^1(X, G_m)
\end{array}
\]
Here, the upper pairing is given by \((x, \mathcal{L}) \mapsto x^*\mathcal{L} = (x, \mathcal{L})^*\mathcal{P}_{\mathcal{A}}\) (the equality by Lemma \([3,5]\) for \( x \in \mathcal{A}(X) \) and \( \mathcal{L} \in \mathcal{A}^t(X) = \operatorname{Pic}^0_{\mathcal{A}/X}(X) \) with \((x, \mathcal{L}) : X \to \mathcal{A} \times_X \mathcal{A}^t\), and the lower pairing is the Yoneda pairing.

**Proof.** The morphism \( \operatorname{Ext}^1_X(\mathcal{A}, G_m) \to \mathcal{A}^t(X) \) is an isomorphism by the Barsotti-Weil formula Theorem \([3,11]\). Given \( x \in \mathcal{A}(X) \), i.e. \( x : X \to \mathcal{A} \), and \( e : (1 \to G_m \to G \to \mathcal{A} \to 0) \in \operatorname{Ext}^1_X(\mathcal{A}, G_m) \), \((x, e)\) maps to \( G \times \mathcal{A} X \) under the Yoneda pairing (composition in the lower row). This is a \( G_m \)-torsor on \( X \), namely \( x^*\mathcal{L} \) if \( G = \mathcal{L} \setminus 0 \) (the zero-section), which is the composition in the upper row.

**Lemma 3.13.** Let \( n \) be invertible on \( X \). Then one has
\[ \mathcal{H}\operatorname{om}_X(\mathcal{A}, G_m) = 0 \quad \text{and} \quad \mathcal{H}\operatorname{om}_X(\mathcal{A}, \mu_n) = 0. \] (3.2)

**Proof.** This holds since \( G_m \) and \( \mu_n \) are affine over \( X \) and \( \mathcal{A}/X \) is proper and has geometrically integral fibres using the Stein factorisation.

**Corollary 3.14.** The low term exact sequence associated to the Ext spectral sequence \( H^n(X, \mathcal{E}\operatorname{xt}^i_X(\mathcal{A}, G_m)) \Rightarrow \operatorname{Ext}^{p+i}_X(\mathcal{A}, G_m) \) (see [Mil80], p. 91, Theorem III.1.22) gives an isomorphism
\[ H^0(X, \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m)) \rightleftharpoons \operatorname{Ext}^1_X(\mathcal{A}, G_m). \] (3.3)

**Proof.** This follows since by (3.2), one has \( E_2^{p,0} = 0 \) for all \( p \) in the Ext spectral sequence.

**Corollary 3.15.** Let \( \ell \) be invertible on \( X \). Then the Ext spectral sequence \( H^n(X, \mathcal{E}\operatorname{xt}^i_X(\mathcal{A}, \mu_{\ell^n})) \Rightarrow \operatorname{Ext}^{p+i}_X(\mathcal{A}, \mu_{\ell^n}) \) gives an injection
\[ H^1(X, \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, \mu_{\ell^n})) \hookrightarrow \operatorname{Ext}^1_X(\mathcal{A}, \mu_{\ell^n}). \] (3.4)

**Proof.** This follows since \( \mathcal{H}\operatorname{om}_X(\mathcal{A}, \mu_{\ell^n}) = 0 \) by (3.2), so \( E_2^{p,0} = 0 \) for all \( p \) in the Ext spectral sequence.

**Lemma 3.16.** Let \( \ell \) be invertible on \( X \). Then one has
\[ \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, \mu_{\ell^n}) = \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m)[\ell^n] = \mathcal{A}^t[\ell^n]. \]

**Proof.** One has a short exact sequence of sheaves
\[ 0 \to \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m)[\ell^n] \to \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m) \xrightarrow{\ell^n} \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m) \to 0 \] (3.5)
since one can check \( \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m)[\ell^n] = \mathcal{A}^t[\ell^n] = 0 \) on stalks by the exactness of the Kummer sequence.

The short exact Kummer sequence yields by (3.2) a short exact sequence
\[ \mathcal{H}\operatorname{om}_X(\mathcal{A}, G_m) = 0 \to \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, \mu_{\ell^n}) \to \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m) \xrightarrow{\ell^n} \mathcal{E}\operatorname{xt}^1_X(\mathcal{A}, G_m) \to 0 \] (3.6)
(the 0 at the right hand side by (3.5)).

Combining (3.5) and (3.6), one gets the first equation in Lemma 3.16. The second equation follows from the Barsotti-Weil formula Theorem 3.11.
**Lemma 3.17.** Let \( \ell \) be invertible on \( X \). Then one has an isomorphism
\[
\delta : \mathcal{H}om_X(\mathcal{O}^n, \mu_{\ell^n}) \xrightarrow{\sim} \mathcal{E}xt^1_X(\mathcal{O}, \mu_{\ell^n}) = \mathcal{O}^1[\ell^n].
\]

**Proof.** Applying the functor \( \mathcal{H}om_X(\mathcal{O}, \mu_{\ell^n}) \) to the short exact Kummer sequence \( 0 \rightarrow \mathcal{O}^n \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0 \) gives an exact sequence
\[
0 \rightarrow \mathcal{H}om_X(\mathcal{O}, \mu_{\ell^n}) \rightarrow \mathcal{H}om_X(\mathcal{O}^n, \mu_{\ell^n}) \rightarrow \mathcal{E}xt^1_X(\mathcal{O}, \mu_{\ell^n}) \rightarrow \mathcal{E}xt^1_X(\mathcal{O}^n, \mu_{\ell^n}).
\]
But multiplication by \( \ell^n \) kills \( \mathcal{E}xt^1_X(\mathcal{O}, \mu_{\ell^n}) \), so the last arrow is zero. Hence \( \delta \) is an isomorphism.

The equality \( \mathcal{E}xt^1_X(\mathcal{O}, \mu_{\ell^n}) = \mathcal{O}^1[\ell^n] \) is Lemma 3.16

**Lemma 3.18.** The diagram
\[
\begin{array}{c}
\mathcal{O}(X)_{n-tors} \otimes \mathbb{Z}_\ell \times \mathcal{O}^1(X)_{n-tors} \otimes \mathbb{Z}_\ell \longrightarrow \text{Pic}(X)_{n-tors} \otimes \mathbb{Z}_\ell
\end{array}
\]
commutes.

**Proof.** The pairing in the lower row identifies with \( H^0(X, \mathcal{O}) \times \text{Ext}^1_X(\mathcal{O}, G_m) \rightarrow H^1(X, G_m) \) by Proposition 3.12.

In the rest of the proof, we show that the following diagram commutes:
\[
\begin{array}{c}
\begin{array}{ccc}
H^1(X, \mathcal{O}^n) & \times & H^1(X, \mathcal{O}^1[\ell^n]) \\
\delta & \downarrow & \delta' \\
H^0(X, \mathcal{O}) & \times & H^0(X, \mathcal{O}^1) \\
& \downarrow & \downarrow \delta' \\
& H^1(X, G_m) & H^1(X, G_m)
\end{array}
\end{array}
\]
(3.7)

Here, the pairing in the upper line is induced by the Weil pairing, and the pairing in the lower line is given by Proposition 3.12. The morphism \( \delta' \) is the connecting morphism of the Kummer sequence. Since \( H^1(X, \mathcal{O}^n) \) is killed by \( \ell^n \), \( \delta \) factors through \( \delta_{\ell^n} \), and analogously for \( \delta' \) and \( \delta' \).

By Proposition 3.12, the pairing \( \mathcal{O}(X) \times \mathcal{O}^1(X) \rightarrow \text{Pic}(X) \) identifies with
\[
H^0(X, \mathcal{O}) \times \text{Ext}^1_X(\mathcal{O}, G_m) \rightarrow H^1(X, G_m).
\]

The diagram
\[
\begin{array}{c}
\begin{array}{ccc}
H^0(X, \mathcal{O}) & \times & \text{Ext}^1_X(\mathcal{O}, G_m) \\
& \downarrow & \downarrow \delta_{\text{ext}} \\
H^0(X, \mathcal{O}) & \times & \text{Ext}^2_X(\mathcal{O}, \mu_{\ell^n}) \\
& \downarrow & \downarrow \delta' \\
& H^2(X, G_m) & H^2(X, G_m)
\end{array}
\end{array}
\]
commutes, where the horizontal maps are Yoneda Ext-pairings, by the \( \delta \)-functoriality \cite{AK70}, p. 67, Theorem 1.1, so we are left with proving that the lower pairing of this diagram and the upper pairing of the diagram (3.7) are equal. In order to show this, we prove the commutativity of
\[
\begin{array}{c}
\begin{array}{ccc}
H^0(X, \mathcal{O}) & \times & \text{Ext}^1_X(\mathcal{O}, \mu_{\ell^n}) \\
& \downarrow & \downarrow \delta \\
H^1(X, \mathcal{O}^n) & \times & H^1(X, \mathcal{O}^1[\ell^n]) \\
& \downarrow & \downarrow \delta' \\
& H^1(X, \mathcal{O}^1[\ell^n]) & H^1(X, \mathcal{O}^1[\ell^n])
\end{array}
\end{array}
\]
(3.8)

\[
\begin{array}{c}
\begin{array}{ccc}
H^1(X, \mathcal{O}^n) & \times & H^2(X, \mathcal{O}^1) \\
& \downarrow & \downarrow \delta' \\
& H^2(X, G_m) & H^2(X, G_m)
\end{array}
\end{array}
\]

note that \( \mathcal{E}xt^1_X(\mathcal{O}, \mu_{\ell^n}) = \mathcal{O}^1[\ell^n] \) by Lemma 3.16.
By adjunction, rewrite the diagram (3.8) as

\[
\begin{array}{ccc}
\Ext^2_X(\mathcal{A}, \mu_{\ell^n}) & \longrightarrow & \Hom(H^0(X, \mathcal{A}), H^2(X, \mu_{\ell^n})) \\
\delta & \downarrow & \\
H^1(X, \mathcal{E}xt^1_X(\mathcal{A}, \mu_{\ell^n})) & \longrightarrow & \Hom(H^1(X, \mathcal{A}[\ell^n]), H^2(X, \mu_{\ell^n})).
\end{array}
\]

Now, the long term exact sequence associated to the local-to-global Ext spectral sequence gives an embedding \( H^1(X, \mathcal{H}om_X(\mathcal{A}[\ell^n], \mu_{\ell^n})) \hookrightarrow \Ext^1_X(\mathcal{A}[\ell^n], \mu_{\ell^n}) \). But by Lemma 3.17 one has an isomorphism \( \delta_1 : H^1(X, \mathcal{H}om_X(\mathcal{A}[\ell^n], \mu_{\ell^n})) \longrightarrow H^1(X, \mathcal{E}xt^1_X(\mathcal{A}, \mu_{\ell^n})) \). Now, the square in the diagram

\[
\begin{array}{ccc}
\Ext^2_X(\mathcal{A}, \mu_{\ell^n}) & \longrightarrow & \Hom(H^0(X, \mathcal{A}), H^2(X, \mu_{\ell^n})) \\
\delta & \downarrow & \\
\Ext^1_X(\mathcal{A}[\ell^n], \mu_{\ell^n}) & \longrightarrow & \Hom(H^1(X, \mathcal{A}[\ell^n]), H^2(X, \mu_{\ell^n})) \\
\hline
H^1(X, \mathcal{E}xt^1_X(\mathcal{A}, \mu_{\ell^n})) & \longrightarrow & \Hom(H^1(X, \mathcal{A}[\ell^n]), H^2(X, \mu_{\ell^n}))
\end{array}
\]

commutes by \( \delta \)-functoriality [AK70], p. 67, Theorem 1.1. The lower triangle commutes by definition and the upper left triangle by functoriality of the Grothendieck spectral sequence and its lower term exact sequence applied to the special case of the local-to-global Ext spectral sequences

\[
\begin{align*}
E_{2,0}^p & = H^p(X, \mathcal{E}xt^q_X(\mathcal{A}, \mu_{\ell^n})) = \Ext^{p+q}_X(\mathcal{A}, \mu_{\ell^n}) \\
E_{2,0}' & = H^p(X, \mathcal{E}xt^q_X(\mathcal{A}[\ell^n], \mu_{\ell^n})) = \Ext^{p+q}_X(\mathcal{A}[\ell^n], \mu_{\ell^n})
\end{align*}
\]

defined on derived categories with edge maps \( \kappa_1, \kappa_1' \) and the exact triangle \( \mathcal{A}[\ell^n] \to \mathcal{A} \to \mathcal{A} \to \mathcal{A}[\ell^n][1] \) inducing \( \mathcal{H}om_X(\mathcal{A}, \mu_{\ell^n}) \xrightarrow{\mathcal{H}om_X(\mathcal{A}[\ell^n], \mu_{\ell^n})} \mathcal{H}om_X(\mathcal{A}[\ell^n][1], \mu_{\ell^n}) \):

\[
\begin{array}{c}
E_{2,1}^{1,1} \xrightarrow{\kappa_1} E^2 \\
\delta \downarrow \quad \delta \\
E_{2,0}' \xrightarrow{\kappa_1'} E^1
\end{array}
\]

Lemma 3.19. The diagram

\[
\begin{array}{ccc}
H^2(X, \mathbb{Z}_\ell(1))_{n-tors} & \xrightarrow{\cup n^{-1}} & H^2 d(X, \mathbb{Z}_\ell(d))_{n-tors} \\
\downarrow \delta_2 = cl^X_2 & & \downarrow cl^X_2 \\
\text{CH}^1(X)_{n-tors} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1)^{d-1} & \xrightarrow{\cup n^{-1}} & \text{CH}^2(X)_{n-tors} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \\
\end{array}
\]

commutes.

Proof. Since the category of \( \mathbb{Z}_\ell \)-modules modulo the Serre subcategory of torsion \( \mathbb{Z}_\ell \)-modules is equivalent to the category of \( \mathbb{Q}_\ell \)-modules, see [Sta15 Tag 0B0K], we can prove the statement after tensoring with \( \mathbb{Q}_\ell \). There is a ring homomorphism

\[
cl^X : \bigoplus_{i=0}^d \text{CH}^i(X) \to \bigoplus_{i=0}^d H^{2i}(\bar{X}, \mathbb{Q}_\ell(i)),
\]

see [Mil80], p. 270, Proposition VI.9.5 (intersection product on the Chow ring and cup product on the cohomology ring) or [Jan88], p. 243, Lemma (6.14), and \( \text{CH}^0(X) \to H^{2d}(\bar{X}, \mathbb{Q}_\ell(d)) \xrightarrow{\phi_{\text{deg}}} \mathbb{Q}_\ell \) maps the class of a point to 1, see [Mil80], p. 276, Theorem VI.11.1 (a).
Comparison of the Yoneda pairing in (3) with the generalised Bloch pairing. This is a generalisation of [ScuS2a] and [BloS2a].

The generalised Bloch pairing

\[ h : A(K) \times A^t(K) \to \mathbb{R}, \]

is defined as follows: Let \( K = k(X) \) be the function field of \( X \). Define the adele ring of \( X \) as the restricted product

\[ A_{K,S} = \prod_{x \in S} K_x \times \prod_{\nu \in \mathcal{O}_X \setminus S} \mathcal{O}_{X,x} \quad \text{where} \quad S \subseteq X^{(1)} \text{ is finite} \]

\[ A_K = \prod_{x \in X^{(1)}} K_x = \lim_{\substack{\to S}} A_{K,S}, \]

where \( K_x \) is the quotient field of the discrete valuation ring \( \mathcal{O}_{X,x} \) with discrete valuation \( v_x : K_x^* \to \mathbb{Z} \) and absolute value \( | \cdot |_{x} = q^{-\deg_x v_x(1)} \). There is the natural homomorphism (the "logarithmic modulus map"), fixing a closed immersion \( i : X \to \mathbb{P}^N_K \) with the very ample sheaf \( \mathcal{O}_X(1) := i^* \mathcal{O}_{\mathbb{P}^N}(1) \)

\[ l : \mathcal{G}_m(A_K) \to \log q : \mathbb{Z} \subseteq \mathbb{R}, (a_x) \mapsto \sum_{x \in X^{(1)}} \log |a_x|_{x} = -\log q \cdot \sum_{x \in X^{(1)}} \deg_x x \cdot v_x(a_x). \quad (3.9) \]

By the product formula (see [Har88], p. 146, Exercise II.6.2 (d)), \( l(\mathcal{G}_m(K)) = l(K^\times) = 0 \). Scale the image \( \log q \cdot \mathbb{Z} \) such that \( l \) is surjective. The generalised Bloch pairing is now given by Definition 3.25.

Recall that \( d = \dim X \). We want to show that the pairing

\[ \langle \cdot , \cdot \rangle : \mathcal{A}(X) \times \text{Ext}^1_{\text{X}_{\text{tor}}}(\mathcal{A}, \mathcal{G}_m) \xrightarrow{\vee} H^1(X, \mathcal{G}_m) \xrightarrow{\cap \mathcal{O}_X(1)^{d-1}} \text{CH}^d(X) \xrightarrow{\deg} \mathbb{Z} \quad (3.10) \]

(note that the Yoneda pairing \( \vee \) identifies with \( \mathcal{A}(X) \times \mathcal{A}(X) \to \text{Pic}(X) \) by Proposition 3.12) coincides up to a factor \( -\log q \) with the generalised Bloch pairing

\[ h : A(K) \times A^t(K) \to \log q \cdot \mathbb{Z} \subseteq \mathbb{R} \quad (3.11) \]

from above, i.e. multiplying the pairing (3.10) by \( -\log q \) gives the pairing (3.11).

Lemma 3.20. The diagram

\[ \begin{array}{ccc}
A(K) & \times & A^t(K) \\
\downarrow & \cong & \downarrow \quad (-\log q) \\
A(K) & \times & \text{Ext}^1_{\text{X}_{\text{tor}}}(\mathcal{A}, \mathcal{G}_m) \langle \cdot , \cdot \rangle \to \mathbb{Z}
\end{array} \]

commutes.

Lemma 3.21. Let \( (R_i)_{i \in I} \) be a family of rings with \( \text{Pic}(\text{Spec } R_i) = 0 \) for every \( i \in I \). Then \( \text{Pic}(\prod_{i \in I} \text{Spec } R_i) = 0 \).

Proof. Line bundles correspond to \( \mathcal{G}_m \)-torsors, and a torsor is trivial iff it has a section. So let a line bundle on \( R := \prod_{i \in I} \text{Spec } R_i \) be represented by an affine scheme \( X \). One has \( X(R) = \prod_{i \in I} X(R_i) \), and each of the factors has a non-trivial element by \( \text{Pic}(R_i) = 0 \). Hence, the product is non-empty.

Corollary 3.22. The Picard group of the adele ring is trivial.

Proof. By the previous Lemma 3.21 \( \text{Pic}(A_{K,S}) = 0 \) since line bundles on local rings are trivial, and \( \text{Pic}(A_K) = \lim_{\to S} \text{Pic}(A_{K,S}) \) by the compatibility of étale cohomology (\( \text{Pic}(X) = H^1(X, \mathcal{G}_m) \)) with limits, see [Mil80], p. 88 f., Lemma III.1.16.

Let \( a \in A(K) \) and \( a^t = (1 \to \mathcal{G}_m \to \mathcal{X} \to \mathcal{A} \to 0) \in \text{Ext}^1_{\mathcal{X}}(\mathcal{A}, \mathcal{G}_m) = \mathcal{A}(X) = A^t(K) \). By descent theory, \( \mathcal{X} \) is a smooth commutative \( X \)-group scheme, and by Hilbert’s theorem 90, the sequence

\[ 1 \to \mathcal{G}_m(K) \to \mathcal{X}(K) \to \mathcal{A}(K) \to \text{H}^1(K, \mathcal{G}_m) = 0 \quad (3.12) \]
and, by Corollary 3.22

\[ 1 \rightarrow G_m(A_K) \rightarrow \mathcal{E}(A_K) \rightarrow \mathcal{A}(A_K) \rightarrow H^1(A_K, G_m) = \text{Pic}(A_K) = 0 \quad (3.13) \]

are still exact.

**Proposition 3.23** (Adele valued points). Let \( X \) be a \( K \)-scheme of finite type and \( S \subseteq X^{(1)} \) be a finite set of places. Then

\[ \lim_{S' \supseteq S} X_S(A_{K(S')}) = X_S(A_K) = X(A_K) \]

and

\[ X_S(A_K, S) = \prod_{v \in S} X_v(K_v) \times \prod_{v \in S} X_v(O_{X,v}) \]

with the notation from [Conb], p. 6.

**Proof.** See [Conb], p. 6, (3.5.1) and Theorem 3.6. \( \square \)

**Lemma 3.24.** The homomorphism \( l : G_m(A_K) \rightarrow \log q \cdot \mathbb{Z} \subseteq \mathbb{R} \) (3.9) has a unique extension \( l_{a^1} : \mathcal{E}(A_K) \rightarrow \mathbb{R} \), which induces by restriction to \( \mathcal{E}(K) \) a homomorphism

\[ l_{a^1} : A(K) \rightarrow \mathbb{R} \]

since \( l(G_m(K)) = 0 \).

**Proof.** Define \( G_{m}^{1} \) as the kernel of \( l \), and \( \mathcal{E}^{1} \) as

\[ \mathcal{E}^{1} = \{ a \in \mathcal{E}(A_K) : \exists n \in \mathbb{Z}_{\geq 1}, na \in \mathcal{E}^{1} \} \]

the rational saturation of \( \mathcal{E}^{1} \) with

\[ \mathcal{E}^{1} = G_m^{1} \cdot \prod_{v \in X^{(1)}} \mathcal{E}(O_{X,v}) \subseteq \mathcal{E}(A_K). \]

Consider the following commutative diagram (at first without the dashed arrows) with exact rows by (3.12) and (3.13) and injective upper vertical morphisms and exact left column:

\[
\begin{array}{cccccc}
1 & \rightarrow & G_m & \rightarrow & \mathcal{E}^1 & \rightarrow & \mathcal{A}(A_K) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & G_m(A_K) & \rightarrow & \mathcal{E}(A_K) & \rightarrow & \mathcal{A}(A_K) & \rightarrow & 0 \\
\downarrow & & l_{a^1} & & \downarrow & & l_{a^1} & & \\
\log q \cdot \mathbb{Z} = & = & \log q \cdot \mathbb{Z} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & & 0 & & 0 & & \\
\end{array}
\]

For the commutativity of the diagram, it suffices to show that (1) \( G_m^{1} = \mathcal{E}^{1} \cap G_m(A_K) \subseteq \mathcal{E}(A_K) \) and (2) \( \mathcal{E}^{1} \rightarrow \mathcal{A}(A_K) \).

Assertion (1) is true because of the following: One has \( G_m^{1} \subseteq G_m(A_K) \cap \mathcal{E}^{1} \) by definition of \( \mathcal{E}^{1} \) and \( G_m^{1} \). For the other inclusion \( \mathcal{E}^{1} \cap G_m(A_K) \subseteq G_m^{1} \), note that

\[ l\left( \prod_{v \in X^{(1)}} G_m(O_{X,v}) \right) = 0 \]
but \(G_m(A_K)/G^1_m \hookrightarrow R\) is torsion-free, hence the inclusion \(\mathcal{X}^{-1} \cap G_m(A_K) \subseteq G^1_m\).

Assertion (2) is true because of the following: By the long exact sequence associated to the short exact sequence \(1 \to G_m \to \mathcal{X} \to \mathcal{A} \to 0\) and Lemma 3.21, there is a surjection

\[
\mathcal{X} \left( \prod_{x \in X(1)} \mathcal{O}_{X,x} \right) \to \mathcal{A} \left( \prod_{x \in X(1)} \mathcal{O}_{X,x} \right) = \mathcal{A}(A_K),
\]

the latter equality by Proposition 3.23 and the valuative criterion for properness. But obviously

\[
\mathcal{X} \left( \prod_{x \in X(1)} \mathcal{O}_{X,x} \right) \subseteq \mathcal{X}^{-1}.
\]

By the snake lemma, the diagram completed with the dashed arrows is also exact and there exists the sought-for extension \(l_{a^t} : \mathcal{X}(A_K) \to \log q \cdot Z\) of \(l : G_m(A_K) \to \log q \cdot Z\). The homomorphism \(l_{a^t}\) induces by restriction to \(\mathcal{X}(K)\) a homomorphism

\[
l_{a^t} : A(K) \to R,
\]

since \(l(G_m(K)) = 0\) by the product formula.

**Definition 3.25.** Define the generalised Bloch pairing \(\langle \cdot , \cdot \rangle : A(K) \times A^t(K) \to \log q \cdot Z\) as follows: Let \(a \in A(K)\) and \(a^t \in A^t(K)\). Let \((a, a^t)\) be the image of \(a\) under the composition of the maps

\[
A(K) = \mathcal{X}(K)/G_m(K) \to \mathcal{X}(A_K)/G_m(K) \xrightarrow{l_{a^t}} \log q \cdot Z
\]

with \(l_{a^t}\) coming from Lemma 3.24.

(The first identity comes from (3.12). Note that \(G_m(K) \subseteq G^1_m\) by the product formula.)

Note that \(\mathcal{A}(X) = \text{Hom}_{\text{fppf}}(Z, \mathcal{A})\). The Yoneda pairing \(\cup : \text{Hom}_{\text{fppf}}(Z, \mathcal{A}) \times \text{Ext}^1_{\text{fppf}}(\mathcal{A}, G_m) \to H^1(X, G_m)\) maps \((a, a^t)\) to the extension \(a \vee a^t\) defined by

\[
a \vee a^t : \begin{array}{c}
1 \longrightarrow G_m \longrightarrow \mathcal{X} \longrightarrow Z \longrightarrow 0 \\
\end{array}
\]

\[
a^t : \begin{array}{c}
1 \longrightarrow G_m \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \longrightarrow 0.
\end{array}
\]

By composition, one gets an extension

\[
l_{a \vee a^t} : \mathcal{X}(A_K) \to \mathcal{X}(A_K) \xrightarrow{l_{a^t}} R
\]

of \(l : G_m(A_K) \to \mathcal{X}(A_K)\), which induces because of \(l(G_m(K)) = 0\) in the exact sequence \(a \vee a^t\) by restriction to \(\mathcal{X}(K)\) a homomorphism

\[
l_{a \vee a^t} : Z \xrightarrow{a} A(K) \xrightarrow{l_{a^t}} R,
\]

so one obviously has

\[
h(a, a^t) = l_{a^t}(a) = l_{a \vee a^t}(1).
\]

By (3.14) and (3.15)

\[
l_{a^t} \left( \prod_{x \in X(1)} \mathcal{X}(\mathcal{O}_{X,x}) \right) = 0, \quad \text{hence} \quad l_{a \vee a^t} \left( \prod_{x \in X(1)} \mathcal{A}(\mathcal{O}_{X,x}) \right) = 0,
\]

by the diagram (3.16) defining \(a \vee a^t\).
Lemma 3.26. Let \((1 \to G_m \to \mathcal{Y} \to Z \to 0) = e \in \text{Ext}^1_{\text{Zar}}(Z, G_m) = H^1(X, G_m) = \text{Pic} X\) be a torseur representing \(\mathcal{L} \in \text{Pic} X\), and let \(l_c : \mathcal{Y}(A_K) \to \mathbb{R}\) be an extension of \(l\) which vanishes on \(\prod_{x \in X^{(1)}} \mathcal{Y}(\mathcal{O}_{X,x})\). Then one has for the homomorphism \(l_c : Z \to \mathbb{R}\) (since \(l_c(G_m(K)) = l(G_m(K)) = 0\)) defined by restriction to \(\mathcal{Y}(K)\):

\[l_c(1) = -\log q \cdot \deg(\mathcal{L} \cap \mathcal{O}_X(1)^{d-1}),\]

where \(\mathcal{O}_X(1)^{d-1}\) denotes the \((d-1)\)-fold self-intersection of \(\mathcal{O}_X(1)\).

(Note that for every \(e\) there is an extension \(l_e\) as in the Lemma using the diagram (3.16) and Lemma 3.24.)

Proof. Considering \(e\) as a class of a line bundle \(\mathcal{L}\) on \(X\), write \(Y(\mathcal{L}) := V(\mathcal{L}) \setminus \{0\text{-section}\}\) for the \(G_m\)-torseur on \(X\) defined by \(\mathcal{L}\). Then \(e\) is isomorphic to the extension

\[1 \to G_m \to \prod_{n \in \mathbb{Z}} Y(\mathcal{L}^{\otimes n}) \to Z \to 0.\]

For every \(x \in |X|\), choose an open neighbourhood \(U_x \subseteq X\) such that \(1 \in Z\) has a preimage \(s_x \in \mathcal{Y}(U_x)\) (these exist by exactness of the short exact sequence \(e\) of sheaves; note that \(\text{Pic}(X) = H^1_{\text{Zar}}(X, G_m) = H^1_{\text{fpqc}}(X, G_m)\) by [Mil66], p. 124, Proposition III.4.9, so there is indeed such a Zariski neighbourhood, not just an fpqc one). Let further \(s \in \mathcal{Y}(K)\) be a preimage of \(1 \in Z\) (note that \(0 \to G_m(K) \to \mathcal{Y}(K) \to Z(K) \to 0\) is exact by Hilbert 90). Then one has \(s_x^{-1} \cdot s \in G_m(K) = K^\times\). Since \(X\) is Jacobson, the \(U_x\) for \(x \in |X|\) cover \(X\). For every \(x \in X^{(1)}\) choose an \(\tilde{x} \in |X|\) such that \(x \in U_{\tilde{x}}\) and set \(s_{\tilde{x}} = s_\tilde{x}\) and \(U_x = U_{\tilde{x}}\). These define a Cartier divisor as \((s_x^{-1} \cdot s) \cdot (s_{\tilde{x}}^{-1} \cdot s)^{-1} = s_x^{-1} \cdot s_{\tilde{x}} \to 1 - 1 = 0 \in Z\), so one has \((s_x^{-1} \cdot s) \cdot (s_{\tilde{x}}^{-1} \cdot s)^{-1} \in G_m(U_x \cap U_{\tilde{x}})^{\text{by the exactness of}}\) of \(1 \to G_m \to \mathcal{Y} \to Z \to 0\), and \([U_x, s_x^{-1}] = \mathcal{L}\) since

\[\Gamma(U_x, \mathcal{O}_X(\langle x, s_x^{-1} \rangle)) = \{f \in K : f s_x^{-1} \in \Gamma(U_x, \mathcal{O}_X)\} = \Gamma(U_x, \mathcal{L}).\]

One has to compare the line bundle \(\mathcal{L}\) with the \(G_m\)-torseur \(\mathcal{Y}\). Now one calculates

\[l_c(1) = l_c(s) = l_c(G_m(K)) = l(G_m(K)) = 0\]

since \(l_c(G_m(K)) = 0\) and \(s \mapsto 1\)

\[= l_c((s_x^{-1} \cdot s))\]

\[= l_c((\prod_{x \in X^{(1)}} \mathcal{Y}(\mathcal{O}_{X,x})) = 0\]

\[= l((s_x^{-1} \cdot s))\]

\[= -\log q \cdot \sum_{x \in X^{(1)}} \text{deg}_x x \cdot v_x(s_x^{-1} \cdot s)\]

by definition of \(l\).

On the other hand, by the above description of \(e\), since the \((U_x, s_x^{-1} \cdot s)\) define a Cartier divisor on \(X\) with associate line bundle isomorphic to \(\mathcal{L}\), one has for \(\text{deg} : \text{CH}^d(X) \to \mathbb{Z}\)

\[\text{deg}(\mathcal{L} \cap \mathcal{O}_X(1)^{d-1}) = \sum_{x \in X^{(1)}} \text{deg}_x x \cdot v_x(s_x^{-1} \cdot s)\]

since \(\text{deg}_x x = \text{deg}([\{x(x)\} \cap X^{d-1}])\) for a generic hyperplane \(H \hookrightarrow \mathbb{P}^N_x\) and \(\mathcal{O}_X(1) = [H] \in \text{CH}^1(X) = \text{Pic}(X)\).

Combining the formulae gives the claim.

Applying Lemma 3.26 to the above situation \(a \in A(K), a' \in A'(K)\) gives us

\[h(a, a') = l_{a \vee a'}(1)\]

by (3.17)

\[= -\log q \cdot \text{deg}([a \vee a'] \cap \mathcal{O}_X(1)^{d-1})\]

by Lemma 3.26

\[= -\log q \cdot \langle a, a' \rangle\]

by (3.10).

(Note that \(\iota\) and \(\mathcal{O}_X(1)\) occur in \(l\) and thus in \(h\).) This finishes the proof of Lemma 3.20.
Comparison of the generalised Bloch pairing and the generalised Néron-Tate height pairing. Let \( K_v \) be the (completion) of \( K = k(X) \) at \( v \in X^{(1)} \), a local field.

Let \( \Delta \) be a divisor on \( A \) defined over \( K_v \) algebraically equivalent to 0 (this corresponds to \( \text{Ext}_X^1(\mathcal{A}, G_m) = \mathcal{A}'(K_v) = \text{Pic}_{A/k}^0(K_v) \)). The divisor \( \Delta \) corresponds to an extension

\[
1 \to G_m \to \mathcal{I}_\Delta \to \mathcal{A} \to 0 \tag{3.18}
\]

in \( \text{Ext}_X^1(\mathcal{A}, G_m) \). Let \( \mathcal{L}_\Delta \) be the line bundle associated to \( \Delta \). Then \( \mathcal{I}_\Delta = V(\mathcal{L}_\Delta) \cap 0 \) with \( \mathcal{L}_\Delta = \mathcal{O}_A(\Delta) \) as a \( G_m \)-torsor. The extension \( (3.18) \) only depends on the linear equivalence class of \( \Delta \).

Restricting to \( K_v \), the extension \( (3.18) \) is split as a torseur over \( A \setminus |\Delta| \) (since a line bundle associated to a divisor \( \Delta \) is trivial on \( X \setminus \Delta \)) by \( \sigma_{\Delta,v}: A \setminus |\Delta| \to \mathcal{I}_\Delta(K_v) \) with \( \sigma_\Delta \) canonical up to translation by \( G_m(K_v) \) (since the choice of \( \sigma_{\Delta,v} \) is the same as the choice of a rational section of \( \mathcal{L}_\Delta \)). Let \( Z_{\Delta,K_v} \) be the group of zero cycles \( \mathcal{A} = \sum n_i(p_i), p_i \in A(K_v) \), on \( A \) defined over \( K_v \) such that \( \sum n_i \deg p_i = 0 \) and \( \supp \mathcal{A} \subseteq A \setminus |\Delta| \). We get a homomorphism \( \sigma_{\Delta,v}: Z_{\Delta,K_v} \to \mathcal{I}_\Delta(K_v) \) (since \( \mathcal{I}_\Delta \) is a group scheme).

We now prove a local analogue of Lemma 3.24

**Lemma 3.27.** There is a commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
1 & & & & 0 & \\
\downarrow & & & & \downarrow & \\
1 & \rightarrow & \mathcal{O}_{K_v}^\times & \rightarrow & \mathcal{I}_\Delta(\mathcal{O}_{K_v}) & \rightarrow & A(K_v) & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & \\
1 & \rightarrow & K_v^\times & \rightarrow & \mathcal{I}_\Delta(K_v) & \rightarrow & A(K_v) & \rightarrow & 0 \\
\downarrow & & \downarrow l_v & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{Z} & = & \mathcal{Z} & \rightarrow & \mathcal{Z} & \rightarrow & \mathcal{Z} & \rightarrow & \mathcal{Z} \\
0 & & 0 & & 0 & & 0 & & 0 & \\
\end{array}
\]

**Proof.** The map labeled \( l_v \) is the valuation map. The short exact sequence in the middle row is \( (3.18) \) evaluated at \( K_v \), and the short exact sequence in the upper row is \( (3.18) \) evaluated at \( \mathcal{O}_{K_v} \): One is left showing that \( \mathcal{I}_\Delta \to A(K_v) \) is surjective. But this follows from the long exact sequence associated to the short exact sequence of sheaves on \( \mathcal{O}_{K_v} \)

\[
1 \to G_m \to \mathcal{I} \to \mathcal{A} \to 0
\]

and Hilbert’s theorem 90: \( H^1(\text{Spec} \mathcal{O}_{K_v}, G_m) = 0 \) since \( \mathcal{O}_{K_v} \) is a local ring. Further, one has \( \mathcal{A}(K_v) = \mathcal{A}(\mathcal{O}_{K_v}) = A(K_v) \) by the valuative criterion of properness and the Néron mapping property. \( \square \)

Now let \( \psi_{\Delta,v}: \mathcal{I}_\Delta(K_v) \to \mathcal{Z} \) be the map defined in the previous lemma. For \( \mathcal{A} \in Z_{\Delta,K_v} \) define

\[
\langle \Delta, \mathcal{A} \rangle_v := \psi_{\Delta,v}\sigma_{\Delta,v}(\mathcal{A}).
\]

**Theorem 3.28.** Let \( K = K(X) \). Let \( a \in A(K) \) and \( a' \in A'(K) \). Let \( \Delta \) resp. \( \mathcal{A} \) be a divisor algebraically equivalent to 0 defined over \( K \) resp. a zero cycle of degree 0 over \( K \) on \( A \) such that \( |\Delta| = a' \) resp. \( \mathcal{A} \) maps to \( a \). Assume \( \supp \Delta \) and \( \supp \mathcal{A} \) disjoint. Then

\[
\langle a, a' \rangle = \log q \cdot \sum_{v \in X^{(1)}} \langle \Delta, \mathcal{A} \rangle_v
\]

with \( \langle a, a' \rangle \) defined as in Definition 3.25

**Proof.** Let

\[
1 \to G_m \to \mathcal{I}_\Delta \to \mathcal{A} \to 0
\]
be the $\mathbb{G}_m$-torsor represented by $a^\chi$, and $\sigma_{\Delta,v} : Z_{\Delta,K} \to \mathcal{H}_{\Delta}(K)$ be as in the local case. One has to show that the map

$$l_{\Delta} : \mathcal{H}_{\Delta}(A_K) \to \log q \cdot \mathbb{Z}$$

(of the above definition in Lemma 3.24) note that $\mathcal{H}_{\Delta}(A_K) \subseteq \mathcal{H}(A_K)$ coincides with the sum of the local maps

$$\psi_{\Delta,v} : \mathcal{H}_{\Delta}(K_v) \to \mathbb{Z}$$

multiplied by $\log q \cdot \deg_v$ for $v \in X^{(1)}$ defined above.

Consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{G}_m^1 & \longrightarrow & \mathcal{H}_{\Delta}(A_K) / \prod_v \mathcal{H}_{\Delta}(\mathcal{O}_{K_v}) \\
\sum_v \deg_v \psi_{\Delta,v} & \longrightarrow & \mathcal{H}_{\Delta}(A_K) / \mathcal{H}_{\Delta}^1 \\
0 & \longrightarrow & \ker(\sum_v \deg_v \psi_{\Delta,v})
\end{array}$$

One has $\prod_v \mathcal{H}_{\Delta}(\mathcal{O}_{K_v}) \subseteq \mathcal{H}_{\Delta}^1$ and $\mathbb{G}_m^1 \subseteq \mathcal{H}_{\Delta}(A_K)$ by the commutative diagram (3.27) in Lemma 3.24, so the exactness of the upper row follows. The exactness of the lower row is clear.

The left square commutes obviously. The right square commutes since by Lemma 3.24 the extension of $l$ to $\mathcal{H}(A_K)$ is unique and $\sum_v \deg_v \psi_{\Delta,v}$ is well-defined (since $\psi_{\Delta,v}$ vanishes on $\mathcal{H}_{\Delta}(\mathcal{O}_{K_v})$) and in the adele ring, almost all components lie in $\mathcal{H}_{\Delta}(\mathcal{O}_{K_v})$ and restricts to $l : \mathbb{G}_m(A_K) \to \mathbb{R}$ since $\psi_{\Delta,v}$ restricts to $l_v$ by Lemma 3.24.

Now let $x \in \mathcal{H}_{\Delta}(A_K) / \mathcal{H}_{\Delta}^1$ with $l_{\Delta}(x) = h$. Lift it to $\tilde{x} \in \mathcal{H}_{\Delta}(A_K) / \prod_v \mathcal{H}_{\Delta}(\mathcal{O}_{K_v})$. Then $\log q \cdot \sum_v \deg_v \psi_{\Delta,v}(\tilde{x}) = h$ by commutativity of the right square. If one chooses another lift, their difference comes from $d \in \mathbb{G}_m^1$, which has height 0, so $\log q \cdot \sum_v \deg_v \psi_{\Delta,v}(\tilde{x})$ only depends on $x$.

**Proposition 3.29.** The local pairings $\langle \Delta, \mathfrak{A} \rangle_v$ coincide with the local Néron height pairings $\langle \Delta, \mathfrak{A} \rangle_{\text{Néron},v}$.

**Proof.** This follows from Néron’s axiomatic characterisation in [BG06], p. 304f., Theorem 9.5.11, which holds for $\langle \Delta, \mathfrak{A} \rangle_v$ by the same argument as in [Bjo80], p. 73ff., (2.11)–(2.15). 

**Corollary 3.30.** The generalised Bloch pairing coincides with the canonical Néron-Tate height pairing.

**Proof.** This is clear since the local Néron-Tate height pairings sum up to the canonical Néron-Tate height pairing, see [BG06], p. 307, Corollary 9.5.14.

**Proof of Theorem 3.3.** Diagram (1) commutes by Lemma 3.10, Diagram (2) commutes by Lemma 3.18 and by Lemma 3.19, Diagram (3) commutes by Lemma 3.20 and by Corollary 3.30.

This finishes the proof of Theorem 3.1.

**Remark 3.31.** Note that the cohomological pairing $(\cdot, \cdot)_\ell$ does not depend on an embedding $\ell : X \hookrightarrow \mathbb{P}^N_\mathbb{Z}$, but all other pairings in the big diagram in Theorem 3.3 depend on a line bundle $\theta$ or cohomology class $\eta \in H^2(X, \mathbb{Z}_\ell(1))$, which manifests in the integral hard Lefschetz defect in the commutative square (1). The two choices, in the integral hard Lefschetz defect in the commutative square (1) and in the other pairings, cancel. Note that in the case of Corollary 3.8 one can choose $\eta$ such that the integral hard Lefschetz defect is trivial.
3.2 The pairing (·, ·)\(_{\ell}\)

**Lemma 3.32.** Assume \(\text{III}(\mathcal{A}/X)[\mathcal{E}]\) is finite. Then one has a commutative diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & & & \\
0 & (\mathcal{A}^t(X) \otimes \mathbb{Z}_\ell)_{\text{n-tors}} & \cong & H^1(X, T_\ell \mathcal{A}^t)_{\text{n-tors}} \\
\downarrow & & & \downarrow \cong (\cup y)^{-1} \\
\mathcal{A}^t(X) \otimes \mathbb{Q}_\ell & \cong & H^1(X, V_\ell \mathcal{A}^t) & \cong (\cup y)^{-1} \\
\downarrow & & & \\
\mathcal{A}^t(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \cong & H^1(X, \mathcal{A}^t[\mathcal{E}]_{\text{div}}) & \cong (\cup y)^{-1} \\
\downarrow & & & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

with the cokernel of \(H^1(X, T_\ell \mathcal{A}^t)_{\text{n-tors}} \hookrightarrow H^{2d-1}(X, T_\ell \mathcal{A}^t)(d-1)_{\text{n-tors}}\) being finite.

**Proof.** The upper left arrow is an isomorphism by Lemma 2.41 For the lower left arrow being an isomorphism: By Lemma 2.30 one has a short exact sequence

\[0 \to \mathcal{A}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^1(X, \mathcal{A}[\mathcal{E}]) \to H^1(X, \mathcal{A})[\mathcal{E}] \to 0.\]

Since \(\mathcal{A}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell\) is divisible, one gets an inclusion \(\mathcal{A}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow H^1(X, \mathcal{A}[\mathcal{E}]_{\text{div}}\). Since \(\text{III}(\mathcal{A}/X)[\mathcal{E}] = H^1(X, \mathcal{A}[\mathcal{E}]_{\text{div}}\) has finite order and is divisible, so it is 0, hence it comes from \(\mathcal{A}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell\).

The upper and middle right arrows are induced by the integral hard Lefschetz theorem Theorem 3.5 (injective) and the hard Lefschetz theorem Theorem 3.4 (isomorphism), respectively, and the lower one by functoriality of the coker-functor. So the lower one surjective by the snake lemma.

For the exactness of the columns: Left column: This column arises from tensoring

\[0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0\]

with \(\mathcal{A}(X)_{\text{n-tors}} \cong \mathbb{Z}^{rk} \mathcal{A}^t(X)\) over \(\mathbb{Z}\). (By the theorem of Mordell-Weil Theorem 2.35 and the Néron mapping property Theorem 2.36 \(\mathcal{A}(X)\) is a finitely generated Abelian group). Middle and right column: This follows from Lemma 2.33. \(\blacksquare\)

**Lemma 3.33.** The homomorphisms induced by the commutative diagram [3.19]

\[
\begin{array}{c}
\text{Hom}(H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{n-tors}}, \mathbb{Z}_\ell) \to \text{Hom}((\mathcal{A}^t(X) \otimes \mathbb{Z}_\ell)_{\text{n-tors}}, \mathbb{Z}_\ell) \quad \text{and} \\
\text{Hom}(H^{2d-1}(X, \mathcal{A}^t[\mathcal{E}](d-1))_{\text{div}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \text{Hom}(\mathcal{A}^t(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathbb{Q}_\ell/\mathbb{Z}_\ell)
\end{array}
\]

are injective with with finite cokernels of the same order (even isomorphic).

**Proof.** Write

\[
\begin{array}{cccc}
0 & \to & A' & \to & A & \to & A'' & \to & 0 \\
\downarrow & & \downarrow f & & \cong & & \downarrow g & & \\
0 & \to & B' & \to & B & \to & B'' & \to & 0
\end{array}
\]

in short for the big diagram [3.19].

The snake lemma gives us \(\ker(g) \sim \ker(f)\) since the middle vertical arrow in [3.20] is an isomorphism. Applying \(\text{Hom}(\cdot, \mathbb{Z}_\ell)\) to the short exact sequence \(0 \to A' \to B' \to \ker(f) \to 0\) gives

\[0 \to \text{Hom}(\ker(f), \mathbb{Z}_\ell) \to \text{Hom}(B', \mathbb{Z}_\ell) \to \text{Hom}(A', \mathbb{Z}_\ell) \to \text{Ext}^1(\ker(f), \mathbb{Z}_\ell) \to \text{Ext}^1(B', \mathbb{Z}_\ell).\]
Since $\ker(f)$ is finite, the first term vanishes and $\Ext^1(coker(f), Z_\ell) \cong coker(f)$, and since $B'$ is torsion-free and finitely generated, hence projective, the last term vanishes. So $\Hom(f, Z_\ell)$ is injective with finite cokernel isomorphic to $coker(f)$.

Applying the exact functor $\Hom(-, Q_\ell/Z_\ell)$ ($Q_\ell/Z_\ell$ is divisible, hence injective) to the short exact sequence $0 \to \ker(g) \to A'' \to B'' \to 0$ gives

$$0 \to \Hom(B'', Q_\ell/Z_\ell) \to \Hom(A'', Q_\ell/Z_\ell) \to \Hom(\ker(g), Q_\ell/Z_\ell) \to 0$$

and $\Hom(\ker(g), Q_\ell/Z_\ell) \cong \ker(g)$ since $\ker(g) \to coker(f)$ is a finite $\ell$-primary group. So $\Hom(g, Q_\ell/Z_\ell)$ is injective with finite cokernel isomorphic to $\ker(g)$. 

\[ \text{Lemma 3.34.} \text{ One has an isomorphism} \]

$$H^2(X, T_\ell A)_{\text{n-tors}} \iso \Hom(H^{2d-1}(X, A[\ell\infty](d-1))_{\text{div}}, Q_\ell/Z_\ell)$$

\[ (3.21) \]

\[ \text{induced by the cup product.} \]

\[ \text{Proof.} \text{ Poincaré duality for the absolute situation} \text{, p. 183, Corollary V.2.3} \text{ (easily generalised to higher dimensions) gives non-degenerate pairings of finite groups for all} \]

$$H^2(X, A[\ell\infty]) \times H^{2d-1}(X, A[\ell\infty](d-1)) \to Q_\ell/Z_\ell.$$  

This is the same as isomorphisms

$$H^2(X, A[\ell\infty]) \iso \Hom(H^{2d-1}(X, A[\ell\infty](d-1)), Q_\ell/Z_\ell),$$

and passing to the projective limit gives us an isomorphism

$$H^2(X, T_\ell A) \iso \Hom(H^{2d-1}(X, A[\ell\infty](d-1)), Q_\ell/Z_\ell).$$

Write $M = H^2(X, T_\ell A)$ and $N = H^{2d-1}(X, A[\ell\infty](d-1))$, so one has $M \iso N^D$. These are finitely and cofinitely generated, respectively. One has

$$M_{\text{n-tors}} = N^D/\lim_n N^D[\ell^n] = N^D/\lim_n N[\ell^n]^D = N^D/\tilde N^D$$

since $0 \to N_{\text{div}} \to N \to \tilde N$ is exact with $\tilde N$ the $\ell$-adic completion of $N$. As $N \cong (Q_\ell/Z_\ell)^r \oplus T$ with $T$ finite, one has $\tilde N \cong T$ (since the $\ell$-adic completion of the divisible group $Q_\ell/Z_\ell$ is trivial) and $h$ surjective. Dualising gives $0 \to \tilde N^D \to N^D \to N_{\text{div}} \to 0$, so $N^D/\tilde N^D = N^D_{\text{div}}$. Summing up, we get $M_{\text{n-tors}} = N^D_{\text{div}}$. 

\[ \text{Theorem 3.35.} \text{ Let} \ III(\mathcal{O}/X)[\ell\infty] \text{ be finite. Then one has det}(-, \cdot)_{\ell} \equiv 1 \text{ for the pairing} (-, \cdot)_{\ell} : H^2(X, T_\ell A)_{\text{n-tors}} \times H^{2d-1}(X, T_\ell A[\ell\infty](d-1))_{\text{n-tors}} \to H^{2d+1}(X, Z_\ell(d)) = Z_\ell \text{ from} \ (2.18). \]

\[ \text{Proof.} \text{ Consider the commutative diagram} \]

\[ \begin{array}{ccc}
\Hom(H^{2d-1}(X, T_\ell A[\ell\infty](d-1))_{\text{n-tors}}, Z_\ell) & \iso & \Hom((A[\ell\infty] \otimes Z_\ell)_{\text{n-tors}}, Z_\ell) \\
\Hom(H^{2d-1}(X, A[\ell\infty](d-1))_{\text{div}}, Q_\ell/Z_\ell) & \iso & \Hom(A[\ell\infty] \otimes Q_\ell/Z_\ell, Q_\ell/Z_\ell).
\end{array} \tag{3.22} \]

The horizontal maps are injective with cokernels finite of the same order by Lemma 3.33.

The right vertical map is an isomorphism: A homomorphism $h : Z_\ell \to Z_\ell$ induces a morphism $Q_\ell \to Q_\ell$ by tensoring with $Q$ and hence a morphism between the cokernels $Q_\ell/Z_\ell \to Q_\ell/Z_\ell$. This is an isomorphism since, by the Mordell-Weil theorem and the Néron mapping property Theorem 2.3, $A[\ell\infty] = Q_\ell/Z_\ell \cong (Q_\ell/Z_\ell)^{\kappa_h} A[\ell\infty]$, and since $H^{2d-1}(X, A[\ell\infty](d-1)) \cong (Q_\ell/Z_\ell)^{\kappa_h} A[\ell\infty]$, and since $\Hom(Q_\ell/Z_\ell, Q_\ell/Z_\ell) = (Q_\ell/Z_\ell)^D = (\lim \frac{Q_\ell}{Z_\ell})^D = \lim \frac{Z_\ell}{\ell^n Z_\ell} = Z_\ell$.
It follows from Poincaré duality for the absolute situation [Mil80], p. 183, Corollary V.2.3 that one has a non-degenerate pairing
\[ (\cdot, \cdot)_\ell : H^2(X, T_\ell A)_{n\text{-tors}} \times H^{2d-1}(X, T_\ell (A^\ell)(d-1))_{n\text{-tors}} \to H^{2d+1}(X, \mathbb{Z}_\ell(d)) = \mathbb{Z}_\ell, \]
so the upper left vertical arrow in (3.22)
\[ H^2(X, T_\ell A)_{n\text{-tors}} \to \text{Hom}(H^{2d-1}(X, T_\ell (A^\ell)(d-1))_{n\text{-tors}}, \mathbb{Z}_\ell) \]
is injective with cokernel of order \( \det(\cdot, \cdot)_\ell \). By comparison of the terms in the commutative diagram (3.22) and using that the horizontal morphisms are injective with cokernels of the same order, see Lemma 3.33, it follows that \( \det(\cdot, \cdot)_\ell = 1 \).

4 The case of a constant Abelian scheme

**Lemma 4.1.** Let \( A \) be an Abelian variety over a finite field \( k \), \( X/k \) be a variety and \( \mathcal{A} = A \times_k X \) be a constant Abelian scheme over \( X \).
1. There is an isomorphism \( \mathcal{A}[m] \xrightarrow{\sim} A[m] \times_k X \) of finite flat group schemes resp. of constructible sheaves (for char \( k \nmid m \)) on \( X \).
2. There is an isomorphism \( T_\ell \mathcal{A} = (T_\ell A) \times_k X \) of \( \ell \)-adic sheaves on \( X \) for \( \ell \neq p \).
3. There is an isomorphism of Abelian groups
\[ \mathcal{A}(X) = \text{Mor}_X(X, \mathcal{A}) \xrightarrow{\sim} \text{Mor}_k(X, A), (f : X \to \mathcal{A}) \mapsto \text{pr}_1 \circ f, \]
and under this isomorphism \( \mathcal{A}(X)_{\text{tors}} \) corresponds to the subset of constant morphisms
\[ \mathcal{A}(X)_{\text{tors}} \to \{ f : X \to A \mid f(X) = \{ a \} \} = \text{Hom}_k(k, A) = A(k). \]

**Proof.** 1. Consider the fibre product diagram
\[
\begin{array}{ccc}
A[m] & \xrightarrow{\cdot k} & k \\
\downarrow & & \downarrow 0 \\
A & \xrightarrow{[m]} & A
\end{array}
\]
and apply \(- \times_k X \).
2. This follows from 1 by passing to the inverse limit over \( m = \ell^n, n \in \mathbb{N} \).
3. The inverse is given by \((f : X \to A) \mapsto ((f, \text{id}_X) : X \to A \times_k X = \mathcal{A})\).

For the second statement: If \( f : X \to A \) takes on the constant value \( a, (f, \text{id}_X) \) has finite order \( \text{ord } a \) in \( A(k) \) since \( k \) and thus \( A(k) \) is finite. Conversely, if \( f : X \to \mathcal{A} \) has finite order \( n \), the image of \( \text{pr}_1 \circ f \) lies in the discrete set of \( n \)-torsion points (since \( \text{pr}_1 : A \times_k X \to A \) is a morphism of group schemes), so is constant because \( X \) is connected.

**Corollary 4.2.** Assume \( X \) has a \( k \)-rational point \( x_0 \). Then there is a commutative diagram with exact rows
\[
\begin{array}{ccc}
0 & \to & A(k) & \to & \mathcal{A}(X) & \to & \text{Hom}_k(\text{Alb}_{X/k}, A) & \to & 0 \\
\downarrow {\cong} & & \downarrow {\cong} & & \downarrow {\cong} & & \downarrow {\cong} & & \downarrow {\cong} \\
0 & \to & \mathcal{A}(X)_{\text{tors}} & \to & \mathcal{A}(X) & \to & \mathcal{A}(X)_{\text{n-tors}} & \to & 0,
\end{array}
\]
and
\[ \text{rk } \mathcal{A}(X) = r(f_A, f_{\text{Alb}_{X/k}}), \]
where \( r(f_A, f_B) \) for \( A \) and \( B \) Abelian varieties over a finite field is defined in [Tat66a], p. 138.

**Proof.** The lower row is trivially exact. By the universal property of the Albanese variety (use that \( X \) has a \( k \)-rational point \( x_0 \), one has \( \{ f \in \text{Mor}_k(X, A) \mid f(x_0) = 0 \} = \text{Hom}_k(\text{Alb}_{X/k}, A) \). Thus the upper row is exact. The left hand vertical arrow is an isomorphism because of Lemma 4.1. Now the five lemma implies that the right hand vertical arrow is an isomorphism since it is a well-defined homomorphism: Precompose \( f : \text{Alb}_{X/k} \to A \) with the Abel-Jacobi map \( \varphi : X \to \text{Alb}_{X/k} \) associated to \( x_0 \).

The equality for the rank follows from [Tat66a], p. 139, Theorem 1 (a).
Example 4.3. The rank of the Mordell-Weil group of a constant Abelian variety over a projective space is 0, since there are no non-constant k-morphisms \( P^n_k \rightarrow A \), see [Mil18], p. 107, Corollary 3.9.

Lemma 4.4. Let \( M \) and \( N \) be torsion-free finitely generated \( \mathbb{Z}_l \)-modules, resp. continuous \( \mathbb{Z}_l[\Gamma] \)-modules. Then one has

\[
M \otimes \mathbb{Z}_l N = \text{Hom}_{\mathbb{Z}_l, \text{Mod}}(M^\vee, N)
\]

\[
(M \otimes \mathbb{Z}_l N)^\Gamma = \text{Hom}_{\mathbb{Z}_l[\Gamma], \text{Mod}}(M^\vee, N)
\]

Proof. This is standard. \( \square \)

Lemma 4.5. Let \( \mathcal{A} = A \times_k X \) be a constant Abelian scheme. Then one has \( H^i(Y, T_Y \mathcal{A}) = H^i(Y, \mathcal{O}_Y) \otimes T_Y A \) as \( \ell \)-adic sheaves on the étale site of \( k \).

Proof. This follows from Lemma 4.12 and the projection formula. \( \square \)

Theorem 4.6. Let \( X/k \) be a smooth projective geometrically connected variety with a \( k \)-rational point. Then the reduced Picard variety \( (\text{Pic}^0_{X/k})_{\text{red}} \) is dual to \( \text{Alb}(X) \) and \( \text{Pic}^0_{X/k} \) is reduced if and only if \( \dim \text{Pic}^0_{X/k} = \dim_k H^2_{\text{Zar}}(X, \mathcal{O}_X) \).

Proof. By [Moc12], Proposition A.6 (i) or [FGI+05], p. 289 f., Remark 9.5.25, \( (\text{Pic}^0_{X/k})_{\text{red}} \) is dual to \( \text{Alb}(X) \). By [FGI+05], p. 283, Corollary 9.5.13, the Picard variety is reduced (and then smooth and an Abelian scheme) if and only if equality holds in \( \dim \text{Pic}^0_{X/k} = \dim_k H^2_{\text{Zar}}(X, \mathcal{O}_X) \). \( \square \)

Remark 4.7. The integer \( a(X) := \dim_k H^2_{\text{Zar}}(X, \mathcal{O}_X) - \dim \text{Pic}^0_{X/k} \) is called the defect of smoothness.

Example 4.8. One has e.g. \( a(X) = 0 \) if (b) from Theorem 4.16 below is satisfied. This holds true for \( X \) an Abelian variety, a K3 surface (since \( H^1_{\text{Zar}}(X, \mathcal{O}_X) = 0 \)) or a curve. For examples of non-reduced Picard schemes of smooth projective surfaces see [Li09].

Lemma 4.9. Let \( f : A \rightarrow B \) a homomorphism of an Abelian varieties and \( e_A : T_f A \times T_f A^t \rightarrow \mathbb{Z}_l(1) \) and \( e_B : T_f B \times T_f B^t \rightarrow \mathbb{Z}_l(1) \) be the perfect Weil pairings. Then

\[
e_B(f(a), b) = e_A(a, f^t(b))
\]

for all \( a \in T_f A \) and \( b \in T_f B^t \), i.e. the diagram

\[
\begin{array}{ccc}
T_f A & \times & T_f A^t \\
\downarrow f & & \downarrow f^t \\
T_f B & \times & T_f B^t
\end{array}
\]

\[
\begin{array}{c}
e_A \\
\uparrow e_A \\
\end{array}
\]

\[
\begin{array}{c}
e_B \\
\uparrow e_B \\
\end{array}
\]

commutes.

Proof. See [Mum70], p. 186, (I). \( \square \)

Corollary 4.10. Let \( f : A \rightarrow A \) be an endomorphism of an Abelian variety \( A \). Then

\[
\text{Tr}_{T_f(A)}(f) = \text{Tr}_{T_f(A^t)}(f^t).
\]

Proof. Choosing an isomorphism \( \mathbb{Z}_l(1) \cong \mathbb{Z}_l \), dualising the diagram in Lemma 4.9 and using that the Weil pairing is perfect Theorem 2.12 gives us a commutative diagram

\[
\begin{array}{ccc}
T_f A & \rightarrow & T_f A \\
\downarrow \cong & & \downarrow \cong \\
(T_f A^t)^\vee & \rightarrow & (T_f A)^\vee
\end{array}
\]

Now use that dualising does not change the trace. \( \square \)
Theorem 4.11 (The cohomological and the trace pairing). Let $X/k$ be a smooth projective geometrically connected variety of dimension $d$ with Albanese variety $A$ such that $\text{Pic}_{X/k}$ is reduced. Denote the constant Abelian scheme $B \times_k X/X$ by $\mathcal{A}/X$. Then the trace pairing

$$\text{Hom}_k(A, B) \times \text{Hom}_k(B, A) \xrightarrow{\delta} \text{End}(A) \xrightarrow{T_\ell} \mathbb{Z}$$

tensored with $\mathbb{Z}_\ell$ equals the cohomological pairing from (2.17)

$$\langle \cdot, \cdot \rangle: H^1(X, T_\ell \mathcal{A})_{\text{tors}} \times H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{tors}} \to H^{2d}(X, \mathbb{Z}_\ell(d)) \xrightarrow{\text{pr}_2} H^d(X, \mathbb{Z}_\ell(d)) = \mathbb{Z}_\ell,$$

and this equals by Theorem 5.7 the Néron-Tate canonical height pairing up to the integral hard Lefschetz defect (see Definition 5.6).

Proof. First note that the Kummer sequence on $X$ gives us a short exact sequence

$$1 = G_m(X)/\ell^n \to H^1(X, \mu_{\ell^n}) \to \text{Pic}(X)[\ell^n] \to 0,$$

the first equality since $G_m(X)/\ell^n = k^*/\ell^n = 1$ since $X/k$ is proper and geometrically integral, and passing to the inverse limit over $n$, an isomorphism $H^1(X, \mathbb{Z}_\ell(1)) = T_\ell \text{Pic}(X) = T_\ell \text{Pic}_{X/k}^0$, the latter equality since $T_\ell \text{NS}(X) = 0$ since the Néron-Severi group is finitely generated by [Mi80], p. 215, Theorem V.3.25.

We show the isomorphisms in the left column of the big commutative diagram on page 36. One has

$$H^1(X, T_\ell \mathcal{A})_{\text{tors}} = H^1(X, T_\ell \mathcal{A})^{\Gamma} \quad \text{by (2.9)}$$

$$= (H^1(X, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} (T_\ell A)(-1))^{\Gamma} \quad \text{by Lemma 4.5}$$

$$= (T_\ell \text{Pic}_{X/k}^0 \otimes_{\mathbb{Z}_\ell} (T_\ell A)(-1))^{\Gamma} \quad \text{by the Kummer sequence}$$

$$= \text{Hom}_{\mathbb{Z}[[\ell]] \otimes \text{Mod}} ((T_\ell A)(-1)^\vee, T_\ell \text{Pic}_{X/k}^0) \quad \text{by Lemma 4.4}$$

$$= \text{Hom}_{\mathbb{Z}[[\ell]] \otimes \text{Mod}} (\text{Hom}(T_\ell(A^t), \mathbb{Z}_\ell)^\vee, T_\ell \text{Pic}_{X/k}^{0, t}) \quad \text{by (2.2)}$$

$$= \text{Hom}_{\mathbb{Z}[[\ell]] \otimes \text{Mod}} (T_\ell(A^t), T_\ell \text{Pic}_{X/k}^{0, t})$$

$$= \text{Hom}_{\mathbb{Z}[[\ell]] \otimes \text{Mod}} (T_\ell(A^t), T_\ell \text{Pic}_{X/k}^0) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell$$

by the Tate conjecture [Tat66a].

Note that $H^1(X, T_\ell \mathcal{A})^{\Gamma}$ is torsion-free since $H^1(X, T_\ell \mathcal{A})$ is so, and this holds because of the Künneth formula and since $H^1(X, \mathbb{Z}_\ell(1)) = T_\ell \text{Pic}_{X/k}^0$ is torsion-free. Therefore, in (2.9) $\ker \alpha = H^0(X, T_\ell \mathcal{A})^{\Gamma}$ is the whole torsion subgroup of $H^1(X, T_\ell \mathcal{A})$.

$T_\ell \mathcal{A}$ has weight $-1$ by Theorem 2.15 and $T_\ell(\mathcal{A}^t)(d-1)$ has weight $-1 - 2(d-1) = -2d + 1$ and from (2.4), we have a commutative diagram with exact rows

$$0 \longrightarrow H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{tors}} \overset{\beta}{\longrightarrow} H^{2d}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{tors}} \longrightarrow H^{2d}(X, T_\ell(\mathcal{A}^t)(d-1)^\Gamma) \longrightarrow 0$$

$$0 \longrightarrow H^{2d-2}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{tors}} \overset{\alpha}{\longrightarrow} H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{tors}} \longrightarrow H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1)^\Gamma) \longrightarrow 0,$$

where only the four groups connected by $f$, $\alpha$ and $\beta$ can be infinite by Lemma 2.28 and as in (2.9).

The perfect Poincaré duality pairing

$$H^1(X, \mathbb{Z}_\ell(1)) \times H^{2d-1}(X, \mathbb{Z}_\ell(d-1)) \to H^{2d}(X, \mathbb{Z}_\ell(d)) \xrightarrow{\sim} \mathbb{Z}_\ell$$

(4.2)

identifies $H^{2d-1}(X, \mathbb{Z}_\ell(d-1))$ with $(T_\ell \text{Pic}_{X/k}^0)^\vee$.

We show the isomorphisms in the left column of the big commutative diagram on page 36. One has

$$H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1))_{\text{tors}} = H^{2d-1}(X, T_\ell(\mathcal{A}^t)(d-1))^\Gamma \quad \text{by (2.9)}$$

$$= (H^{2d-1}(X, \mathbb{Z}_\ell(d-1)) \otimes_{\mathbb{Z}_\ell} T_\ell(A^t))^\Gamma \quad \text{by Lemma 4.5}$$

$$= ((T_\ell \text{Pic}_{X/k}^0)^\vee \otimes_{\mathbb{Z}_\ell} T_\ell(A^t))^\Gamma \quad \text{by (4.2)}$$

$$= \text{Hom}_{\mathbb{Z}[[\ell]] \otimes \text{Mod}} (T_\ell \text{Pic}_{X/k}^0, T_\ell(A^t)) \quad \text{by Lemma 4.4}$$

$$= \text{Hom}_{\mathbb{Z}[[\ell]] \otimes \text{Mod}} (\text{Pic}_{X/k}^0, A^t) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell$$

by the Tate conjecture [Tat66a].
Example 4.12. In particular, if the characteristic polynomials of the Frobenius on $\text{Pic}^0_{X/k}$ and $A^t$ are coprime, then $\text{Hom}_k(A^t, \text{Pic}^0_{X/k}) = 0 = \text{Hom}_k(\text{Pic}^0_{X/k}, A^t)$ and the discriminants of the parings $(\cdot, \cdot)$ and $(\cdot, \cdot)$ from (2.17) and (2.18) are equal to 1.

In the general case, we have to investigate if the big diagram on page 36 commutes.

(1) commutes since $\cup$-product commutes with restrictions.

(2) commutes because of the associativity of the $\cup$-product.

(3) commutes since, in general, one has a commutative diagram of finitely generated free modules over a ring $R$

$$
\begin{array}{c}
A \times B \xrightarrow{(\cdot, \cdot)} R \\
\cong \\
C \times C' \xrightarrow{\cong} R
\end{array}
$$

identifying $B$ with the dual of $C \cong A$ with a perfect pairing $(\cdot, \cdot)$ and the canonical pairing $C \times C' \to R$.

(4) commutes since, in general, one has a commutative diagram of finitely generated free modules over a ring $R$

$$
\begin{array}{c}
(M \otimes_R N') \times (M' \otimes_R N) \xrightarrow{\cong} \text{End}_{R-\text{Mod}}(M) \otimes_R \text{End}_{R-\text{Mod}}(N) \\
\text{Hom}_{R-\text{Mod}}(N, M) \times \text{Hom}_{R-\text{Mod}}(M, N) \xrightarrow{\cong} \text{End}_{R-\text{Mod}}(N) \xrightarrow{\text{Tr}_N} R
\end{array}
$$

(5) commutes because of precomposing with the isomorphism $(T_{1, A'}(-1))^\vee \xrightarrow{\sim} T_1(A')$ coming from the perfect Weil pairing Theorem 2.12.

(6) commutes because of [Lan58, p. 186f., Theorem 3].

(7) commutes because of $\text{Tr}(\alpha \beta) = \text{Tr}(\beta \alpha)$, see [Lan58, p. 187, Corollary 1]. (The morphism $\sigma$ switches the two factors.)

(8) commutes because of Corollary 4.10 and since $\text{Pic}^0_{X/k}$ is dual to $\text{Alb}(X)$ by Theorem 4.6 since $\text{Pic}_{X/k}$ is reduced.

This finishes the proof of Theorem 4.11.

The case of a curve as a basis. Let $X/k$ be a smooth projective geometrically connected curve with function field $K = k(X)$, base point $x_0 \in X(k)$, Albanese variety $A$, Abel-Jacobi map $\varphi : X \to A$ with canonical principal polarisation $\epsilon : A \to A^t$, and $B/k$ be an Abelian variety.

Let $(\cdot, \cdot) : \text{Hom}_k(A, B) \times \text{Hom}_k(B, A) \to \mathbb{Z}, (\alpha, \beta) \mapsto (\alpha, \beta) := \text{Tr}(\beta \circ \alpha : A \to A) \in \mathbb{Z}$

be the trace pairing, the trace being taken as an endomorphism of $A$ as in [Lan58]. By [Lan58, p. 186f., Theorem 3], this equals to the trace taken as an endomorphism of the Tate module $T_1 A$ or $H^1(A, \mathbb{Z}_l)$ (they are dual to each other by Theorem 2.13), so for the trace, it does not matter which one we are taking.

We now show that our trace pairing is equivalent to the usual Néron-Tate height pairing on curves and is thus a sensible generalisation to the case of a higher dimensional base.

Proposition 4.13. Let $X, Y$ be Abelian varieties over a field $k$ and $f \in \text{Hom}_k(X, Y)$. Then

$$(f \times \text{id}_{Y^t})^* \mathcal{P}_Y \cong (\text{id}_X \times f)^* \mathcal{P}_X$$

in $\text{Pic}(X \times_k Y^t)$.

Proof. By the universal property of the Poincaré bundle $\mathcal{P}_X$ applied to $(f \times \text{id}_{\mathcal{P}_Y})^* \mathcal{P}_Y$, there exists a unique map $\hat{f} : X^t \to Y^t$ such that

$$(f \times \text{id}_{Y^t})^* \mathcal{P}_Y \cong (\text{id}_X \times \hat{f})^* \mathcal{P}_X. \quad (4.3)$$

It remains to show that $\hat{f} = f^t$. 
Note that \((T_\ell A(-1))^\vee = T_\ell A^t\) in (5) by \([2.2]\).
Let $T/k$ be a variety and $\mathcal{L} \in \text{Pic}^0(Y \times_k T)$ arbitrary. By the universal property of the Poincaré bundle $\mathcal{P}_Y$, there exists $g : T \to Y'$ such that $\mathcal{L} = (\text{id}_Y \times g)^* \mathcal{P}_Y$. We want to show $f_* : Y'(T) \to X'(T), g \mapsto fg$ equals $f' : \text{Pic}^0(Y \times_k T) \to \text{Pic}^0(X \times_k T), \mathcal{L} \mapsto f^* \mathcal{L}$. Now we have

$$f'(\mathcal{L}) = (f \times \text{id}_T)^* \mathcal{L} = (f \times \text{id}_T)^*(\text{id}_Y \times g)^* \mathcal{P}_Y = (f \times g)^* \mathcal{P}_Y = (\text{id}_X \times g)^*(f \times \text{id}_T)^* \mathcal{P}_X \quad \text{by (4.3)}$$

$$= (\text{id}_X \times \hat{g})^* \mathcal{P}_X = f_*(\mathcal{L})$$

$$\exists \text{Proposition 4.14.}$$

Let $\vartheta^-$ be the class of $[-1]^* \Theta$ with the Theta divisor as in [BG06], p. 272, Remark 8.10.8, and $\delta_1 \in \text{Pic}(X \times A)$ as in [BG06], p. 278, l. −4 the Poincaré class. Let $\varphi$ be the Abel-Jacobi map and $\varphi_{\vartheta^-}$ as in [BG06], p. 252, Theorem 8.5.1. Let $c_A = m^* \vartheta^- - \text{pr}_1^* \vartheta^- - \text{pr}_2^* \vartheta^- \in \text{Pic}(A \times_k A)$ with $m : A \times_k A \to A$ the addition morphism and $\text{pr}_1 : A \times_k A \to A$ the projections. Then

$$(\varphi \times \text{id}_A)^* c_A = -\delta_1 \quad (4.4)$$

and

$$(\text{id}_A \times \varphi_{\vartheta^-})^* \mathcal{P}_A = c_A. \quad (4.5)$$

Proof. See [BG06], p. 279, Propositions 8.10.19 and 8.10.20.

$$\exists \text{Theorem 4.15 (The trace and the height pairing for curves).}$$

Let $X/k$ be a smooth projective geometrically connected curve with Albanese variety $A$. Then the trace pairing

$$\text{Hom}_k(A, B) \times \text{Hom}_k(B, A) \xrightarrow{\circ} \text{End}(A) \xrightarrow{\text{tr}} \mathbb{Z}, (\alpha, \beta) \mapsto (\alpha, \beta)$$

equals the following height pairing

$$(\alpha, \beta)_{\text{ht}} := \deg_X(-\gamma(\alpha), \gamma'(\beta))^* \mathcal{P}_B) = \deg_X(-(\alpha \varphi, \beta' \varphi)^* \mathcal{P}_B),$$

where $\varphi : X \to A$ is the Abel-Jacobi map associated to a rational point of $X$, $c : A \sim A^t$ is the canonical principal polarisation associated to the theta divisor, and

$$\gamma(\alpha) : X \xrightarrow{\sim} A \xrightarrow{\alpha} B,$$

$$\gamma'(\beta) : X \xrightarrow{\sim} A \xrightarrow{\beta'} A^t \xrightarrow{\sim} B^t,$$

and this is equivalent to the usual Néron-Tate canonical height pairing.

Proof. By [Mil68], p. 100, we have

$$(\alpha, \beta) = \deg_X((\text{id}_X, \beta \varphi)^* \delta_1),$$

where $\delta_1 \in \text{Pic}(X \times_k A)$ is a divisorial correspondence such that

$$(\text{id}_X \times \varphi)^* \delta_1 = \Delta_X - \{x_0\} \times X - X \times \{x_0\}$$

with the diagonal $\Delta_X \hookrightarrow X \times_k X$, see [BG06], p. 279, Proposition 8.10.18.

Note the property Proposition 4.14 of the Theta divisor $\Theta$ of the Jacobian $A$ of $C$ on $A$ (which is defined in [BG06], p. 272, Remark 8.10.8) and let $\Theta^- = [-1]^* \Theta$ with $\vartheta$ and $\vartheta^-$ denoting the respecting divisor class.
The Theta divisor induces the canonical principal polarisation $\varphi_\theta = c : A \to A^t$. Therefore

$$(\alpha \varphi \times \beta^t \varphi)^* \mathcal{P}_B = (\alpha \varphi \times c \varphi)^*(\text{id}_X \times \beta^t)^* \mathcal{P}_B$$

$$= (\alpha \varphi \times c \varphi)^*(\beta \times \text{id}_A)^* \mathcal{P}_A$$

by Proposition 4.13

$$= (\beta \alpha \varphi \times c \varphi)^* \mathcal{P}_A$$

$$= (\beta \alpha \varphi \times \varphi \beta \varphi)^* \mathcal{P}_A$$

$$= (\alpha \varphi \times \varphi \beta \varphi)^* \mathcal{P}_A$$

by (4.3)

$$= (\varphi \times \beta \alpha \varphi)^* c_A$$

by symmetry of $c_A$

$$= -(\text{id}_X \times \beta \alpha \varphi)^* \delta_1$$

by (4.4)

Summing up, one has

$$(\alpha, \beta)_{ht} = \deg_X \langle -(\text{id}_X, \beta \alpha \varphi)^* \delta_1 \rangle$$

$$= -\langle \alpha, \beta \rangle.$$

By [MBSS], p. 72, Théorème 5.4, this pairing equals the Néron-Tate canonical height pairing.

**Theorem 4.16.** Let $k = \mathbb{F}_q$, $q = p^n$ be a finite field and $X/k$ a smooth projective and geometrically connected variety and assume $X = X \times_k \bar{k}$ satisfies

(a) the Néron-Severi group of $X$ is torsion-free and

(b) the dimension of $H^1(X, \mathcal{O}_X)$ as a vector space over $\bar{k}$ equals the dimension of the Albanese variety of $X/k$.

If $B/k$ is an Abelian variety, then $H^1(X, B)$ is finite and its order satisfies the relation

$$q^{nd} \prod_{a_i \neq b_j} \left(1 - \frac{a_i}{b_j}\right) = |H^1(X, B)| |\det (\alpha_i, \beta_j)|,$$

where $A/k$ is the Albanese variety of $X/k$, $g$ and $d$ are the dimensions of $A$ and $B$, respectively, $(a_i)_{i=1}^{2g}$ and $(b_j)_{j=1}^{2d}$ are the roots of the characteristic polynomials of the Frobenius of $A/k$ and $B/k$, $(\alpha_i)_{i=1}^r$ and $(\beta_i)_{i=1}^s$ are bases for $\text{Hom}_k(A, B)$ and $\text{Hom}_k(B, A)$, and $\langle \alpha_i, \beta_j \rangle$ is the trace of the endomorphism $\beta_j \alpha_i$ of $A$.

**Proof.** See [Mil68], p. 98, Theorem 2.

**Remark 4.17.** Note that $\text{Hom}_k(A, B)$ and $\text{Hom}_k(B, A)$ are free $\mathbb{Z}$-modules of the same rank $r = r(f_A, f_B) \leq 4gd$ by [Tat65a], p. 139, Theorem 1 (a), with $f_A$ and $f_B$ the characteristic polynomials of the Frobenius of $A/k$ and $B/k$. (Another argument for them having the same rank is that the category of Abelian varieties up to isogeny is semi-simple, decomposing $A$ and $B$ into simple factors.) Furthermore, $H^1(X, B) = H^1(X, B \times_k X) = \text{III}(B \times_k X/X)$ since for $U \to X$, one has $B(U) = (B \times_k X)(U)$ by the universal property of the fibre product.

**Remark 4.18.** For the question (a) and (b) in Theorem 4.16 is satisfied, see Example 4.20.

**Lemma 4.19.** Let $k = \mathbb{F}_q$ be a finite field and $A/k$ be an Abelian variety of dimension $g$. Denote the eigenvalues of the Frobenius $\text{Frob}_q$ on $V_t A$ by $(\alpha_i)_{i=1}^{2g}$. Then $\alpha_i \mapsto q^{\frac{\alpha_i}{g}}$ is a bijection.

**Proof.** The Weil pairing induces a perfect Galois equivariant pairing

$$V_t A \times V_t A^t \to \mathbb{Q}_l(1),$$

and, choosing a polarisation $f : A \to A^t$, by Lemma 2.16 we also have by precomposing a perfect Galois equivariant pairing

$$\langle \cdot, \cdot \rangle : V_t A \times V_t A \to \mathbb{Q}_l(1).$$

Now let $v_i$ be an eigenvector of $\text{Frob}_q$ on $V_t A$ with eigenvalue $\alpha_i$. Then, since the pairing $\langle \cdot, \cdot \rangle$ is perfect, there is an eigenvector $v_j$ of $\text{Frob}_q$ on $V_t A$ such that $\langle v_i, v_j \rangle = x \neq 0$ (otherwise, we would have $\langle v_i, v_j \rangle = 0$ for all eigenvectors $v_j$, but there is a basis of eigenvectors on the Tate module since the Frobenius acts semi-simply). Now, since the pairing is Galois equivariant, $q x = \text{Frob}_q(x) = \text{Frob}_q(\langle v_i, v_j \rangle) = \langle \text{Frob}_q v_i, \text{Frob}_q v_j \rangle = \langle \alpha_i v_i, \alpha_j v_j \rangle = \alpha_i \alpha_j \langle v_i, v_j \rangle = \alpha_i \alpha_j x$. Since $x \neq 0$, the statement follows.
Example 4.20. (a) and (b) are satisfied for $X = A$ an Abelian variety, a K3 surface or a curve: (a) because of Mimura, p. 178, Corollary 2 and Huybrechts, p. 364, and (b) for curves and Abelian varieties since $A^f = \text{Pic}^0_{A/k}$ is an Abelian variety, in particular smooth and reduced, and by Example 4.3 for K3 surfaces. See also Theorem 4.6.

Remark 4.21. Define the regulator $R(\mathfrak{A}/X)$ as $|\text{det}(\cdot, \cdot)|$.

By Remark 4.17 we get

**Corollary 4.22.** In the situation of Theorem 4.16 one has

$$q^{rd} \prod_{a_i \neq b_j} \left(1 - \frac{a_i}{b_j}\right) = |\text{III}(B \times_k X/X)| R(\mathfrak{A}/X).$$

**Definition 4.23.** Define the $L$-function of $B \times_k X/X$ as the $L$-function of the Chow motive

$$h^1(B) \otimes (h^0(X) \otimes h^1(X)) = h^1(B) \otimes (h^1(B) \otimes h^1(X)),$$

namely

$$L(B \times_k X/X, s) = \frac{L(h^1(B) \otimes h^1(X), s)}{L(h^1(B), s)}.$$

Here, the Künneth projectors are algebraic by DM91, p. 217, Corollary 3.2.

**Theorem 4.24.** The two $L$-functions Definition 2.2 and Definition 4.23 agree for constant Abelian schemes.

**Proof.** One has $V_t B = H^1(\bar{B}, \mathbb{Q}_t)^\vee$ by Theorem 2.13 ($V_t B)^\vee = (V_t B)^*(-1)$, $V_t B \cong V_t B)$ by Lemma 2.16, and the existence of a polarisation of $\mathbb{M}$, p. 113, Theorem 7.1, $H^1(X, V_t \mathfrak{A}) = H^1(X, V_t) \otimes V_t B$ by Lemma 4.5, and $V_t \mathfrak{A} = (V_t B) \times_k X$ by Lemma 4.1, since $\mathfrak{A}/X$ is constant. Using this, one gets

$$L(h^1(X) \otimes h^1(B), t) = \text{det}(1 - \text{Frob}_q^{-1} t | H^1(X, \mathbb{Q}_t) \otimes H^1(\bar{B}, \mathbb{Q}_t))$$

$$= \text{det}(1 - \text{Frob}_q^{-1} t | H^1(X, \mathbb{Q}_t) \otimes V_t(B)^*(-1))$$

$$= \text{det}(1 - \text{Frob}_q^{-1} t | H^1(X, (V_t B) \times_k X)^*(-1))$$

$$= \text{det}(1 - \text{Frob}_q^{-1} q^{-1} t | H^1(X, V_t \mathfrak{A}))$$

$$= L(\mathfrak{A}/X, q^{-1} t).$$

Now conclude using $h^1(B) = h^0(X) \otimes h^1(B)$ since $X$ is connected.

**Remark 4.25.** Note that

$$\text{ord}_{t=1} L(\mathfrak{A}/X, t) = \text{ord}_{s=1} L(\mathfrak{A}/X, q^{-1} q^s).$$

**Remark 4.26.** Now let us explain how we came up with this definition of the $L$-function. We omit the characteristic polynomials $L_i(\mathfrak{A}/X, t)$ in higher dimensions $i > 1$ since otherwise cardinalities of cohomology groups would turn up in the special $L$-value which we have no interpretation for (as in the case $i = 0$ and the cardinality of the $\ell$-torsion of the Mordell-Weil group, or in the case $i = 1$ and the cardinality of the $\ell$-torsion of the Tate-Shafarevich group). In the case of a curve $C$ as a basis, our definition is the same as the classical definition of the $L$-function up to an $L_2(t)$-factor. This factor contributes basically only a factor $|\mathfrak{A}|^{\ell}_\infty|^{\ell}\text{tors}$ in the denominator. In the classical curve case $\dim X = 1$, the $L$-function can also be represented as a product over all closed points $x \in |X|$ of Euler factors.

We expand

$$L(B \times_k X/X, s) = \frac{L(h^1(B) \otimes h^1(X), s)}{L(h^1(B), s)}$$

$$= \prod_{j=1}^{2d} \prod_{i=1}^{2q_j} (1 - a_i b_j q^{-s})$$

$$= \prod_{j=1}^{2d} (1 - b_j q^{-s}).$$
By Lemma 4.19 one has for the numerator
\[
\prod_{j=1}^{2d} \prod_{i=1}^{2g} (1 - a_i b_j q^{-s}) = \prod_{j=1}^{2d} \prod_{i=1}^{2g} \left( 1 - \frac{a_i}{b_j} q^{1-s} \right),
\]
(4.6)
and the denominator has no zeros at \( s = 1 \) by the Riemann hypothesis (the eigenvalues of the Frobenius \( (b_j) \) on \( h^1(B) \) have absolute value \( q^{1/2} \)). Therefore
\[
\operatorname{ord}_{s=1} L(B \times_k X/X, s) = \tau(f_A, f_B)
\]
is equal to the number \( \tau(f_A, f_B) \) of pairs \((i, j)\) such that \( a_i = b_j \), which equals by [Tat66a], p. 139, Theorem 1 (a) the rank \( r \) of \((B \times_k X)(X)\):
\[
\tau(f_A, f_B) = \operatorname{rk} \operatorname{Hom}_k(A, B) \\
= \operatorname{rk} \operatorname{Hom}_k(X, B) \quad \text{by the universal property of the Albanese variety} \\
= \operatorname{rk} \operatorname{Hom}_X(X, B \times_k X),
\]
see Corollary 4.2

**Lemma 4.27.** The denominator evaluated at \( s = 1 \) equals
\[
\prod_{j=1}^{2d} (1-b_j q^{-1}) = \frac{|(B \times_k X)(X)_{\text{tors}}|}{q^d}.
\]

**Proof.**
\[
\prod_{j=1}^{2d} (1-b_j q^{-1}) = \prod_{j=1}^{2d} (1- \frac{1}{b_j}) \quad \text{by Lemma 4.19} \\
= \prod_{j=1}^{2d} b_j - 1 \\
= \prod_{j=1}^{2d} \frac{1}{b_j} \quad \text{since } 2d \text{ is even} \\
= \deg(\id - \operatorname{Frob}_q) \\
= \frac{|B(\mathbb{F}_q)|}{q^d} \quad \text{since } \id - \operatorname{Frob}_q \text{ is separable} \\
= \frac{|(B \times_k X)(X)_{\text{tors}}|}{q^d} \quad \text{by Lemma 4.13}. \quad \square
\]

**Remark 4.28.** Note that, if \( X/k \) is a smooth curve, \((B \times_k X)(X) = B(K)\) with \( K = k(X) \) the function field of \( X \) by the valuative criterion for properness since \( X/k \) is a smooth curve and \( B/k \) is proper. For general \( X \), setting \( \mathcal{A} = B \times_k X \) and \( K = k(X) \) the function field, \((B \times_k X)(X) = \mathcal{A}(X) = A(K)\) also holds true because of the Néron mapping property.

**Remark 4.29.** One has \(|(B \times_k X)(X)_{\text{tors}}| = |B(k)| = |B^t(k)| = |(B \times_k X)^t(X)_{\text{tors}}|\) by Lemma 4.13 and Lemma 2.21.

Recall Remark 2.42

Putting everything together,
\[
\lim_{s \to 1} \frac{L(\mathcal{A}/X, s)}{(s-1)^r} = \frac{q^d (\log q)^r}{|\mathcal{A}(X)_{\text{tors}}|} \prod_{a_i \neq b_j} (1 - \frac{a_i}{b_j}) \quad \text{by Lemma 4.27 and 4.6} \\
= q^{(g-1)d (\log q)^r} \frac{|\prod (\mathcal{A}/X)| \cdot R(\mathcal{A}/X)}{|\mathcal{A}(X)_{\text{tors}}|} \quad \text{by Corollary 4.22}
\]
Theorem 4.30. In the situation of Theorem 4.16, one has:
1. The Tate-Shafarevich group $\text{III}(\mathcal{A}/X)$ is finite.
2. The vanishing order equals the Mordell-Weil rank $r$: $\text{ord}_{x=1} L(\mathcal{A}/X, s) = \text{rk} \mathcal{A}(X) = \text{rk} A(K)$.
3. There is the equality for the leading Taylor coefficient
   \[ L^*(\mathcal{A}/X, 1) = q^{g-1}d(\log q)^r \frac{\text{III}(\mathcal{A}/X) \cdot R(\mathcal{A}/X)}{\mathcal{A}(X)_{\text{tors}}} \, . \]

Combining Theorem 2.44 and Theorem 4.30 and using Theorem 4.24 one can identify the remaining two expressions in Theorem 2.44.

Corollary 4.31. In the situation of Theorem 4.16 in Theorem 2.44 resp. Lemma 2.47 all equalities hold and one has
   \[ |\det(\cdot, \cdot)|_{\ell}^{-1} = 1, \quad |H^2(X, T_{\mathcal{A}})| = 1. \]

Remark 4.32. For constant Abelian schemes $\mathcal{A} = A \times_k X$ (under the assumption (a) above that $\text{NS}(\bar{X})$ is torsion-free), one has $|H^2(\bar{X}, T_{\mathcal{A}})| = 1$ (this factor turns up in the Birch-Swinnerton-Dyer formula for the special $L$-value Theorem 2.44): The long exact sequence associated to the Kummer sequence yields the exactness of
   \[ 0 \to H^1(\bar{X}, G_m)/\ell^n \to H^2(\bar{X}, \mu_{\ell^n}) \to H^2(\bar{X}, G_m)[\ell^n] \to 0. \]

Combining with the exactness of
   \[ 0 \to \text{Pic}^0(\bar{X}) \to \text{Pic}(\bar{X}) \to \text{NS}(\bar{X}) \to 0 \]
and the divisibility of $\text{Pic}^0(\bar{X})$ (since multiplication by $\ell^n$ on an Abelian variety is an isogeny, hence surjective, by [Mil86a, p. 115, Theorem 8.2]), hence $H^1(\bar{X}, G_m)/\ell^n = \text{Pic}(\bar{X})/\ell^n = \text{NS}(\bar{X})/\ell^n$, and passage to the inverse limit $\lim_{\ell \to \infty}$ gives us
   \[ 0 \to \text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell \to H^2(\bar{X}, \mathbb{Z}_\ell(1)) \to T_1H^2(\bar{X}, G_m) \to 0 \]
since the $\text{NS}(\bar{X})/\ell^n$ are finite by [Mil80, p. 215, Theorem V.3.25], so they satisfy the Mittag-Leffler condition. As $\text{NS}(\bar{X})$ is torsion-free (by assumption (a) above) and $T_1H^2(\bar{X}, G_m)$ too (as a Tate module), it follows that $H^2(\bar{X}, \mathbb{Z}_\ell(1))$ is torsion-free, so also
   \[ H^2(\bar{X}, T_{\mathcal{A}}) = H^2(\bar{X}, \pi^*(T_{\mathcal{A}})) = H^2(\bar{X}, \pi^*(T_{\mathcal{A}}(1)) \otimes \mathbb{Z}_\ell(1)) = (H^2(\bar{X}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Z}_\ell T_{\mathcal{A}}(1)) \]
by Lemma 4.112 (here we are using that $\mathcal{A}/X$ is constant) and the projection formula for $\pi: \bar{X} \to k$ (similar to Lemma 2.18), so
   \[ |H^2(\bar{X}, T_{\mathcal{A}})| = 1 \]
since this group is finite by Corollary 2.26 (having weight $2 - 1 = 1 \neq 0$ by Theorem 2.2 and Theorem 2.15) and torsion-free (as a subgroup of a tensor product of torsion-free finite rank groups).

5 Verification of the conjecture for certain Abelian schemes

We assume in this section that all varieties have a base point. This assumption is needed for Proposition 5.1.

Proposition 5.1. Let $X, Y$ be smooth proper varieties over a field $k$ with a $k$-rational point. Then there is an exact sequence of $k$-group schemes
   \[ 0 \to \text{Pic}_{X/k} \times_k \text{Pic}_{Y/k} \to \text{Pic}_{X \times_k Y/k} \to \text{Hom}(\text{Alb}_{X/k}, \text{Pic}^0_{Y/k}), \]
which is short exact on geometric points.


Corollary 5.2. Let $X, Y$ be smooth proper varieties over an algebraically field $k$ with a $k$-rational point. If $\text{Pic}^0_{X/k}$ and $\text{Pic}^0_{Y/k}$ are reduced, so is $\text{Pic}^0_{X \times_k Y/k} = \text{Pic}^0_{X/k} \times_k \text{Pic}^0_{Y/k}$.
Proof. One has $\text{Pic}^0_{X \times_k Y/k} = \text{Pic}^0_{X/k} \times_k \text{Pic}^0_{Y/k}$ from the exact sequence in Proposition 5.1 by taking the connected component of 0 and since the Hom-scheme is discrete. Now use that the fibre product of reduced varieties over an algebraically closed field is reduced [GW10, p. 135, Proposition 5.49]. □

**Corollary 5.3.** Let $X, Y$ be smooth proper varieties over an algebraically closed field $k$ with a $k$-rational point. If $\text{NS}(X)$ and $\text{NS}(Y)$ are free, so is $\text{NS}(X \times_k Y)$.

**Proof.** By Proposition 5.1 and Corollary 5.2 there is a commutative diagram with exact rows

$$
\begin{array}{c}
0 \rightarrow \text{Pic}^0(X) \times \text{Pic}^0(Y) \xrightarrow{\cong} \text{Pic}^0(X \times_k Y) \rightarrow 0 \\
0 \rightarrow \text{Pic}(X) \times \text{Pic}(Y) \rightarrow \text{Pic}(X \times_k Y) \rightarrow \text{Hom}_k(\text{Pic}^0_{X/k}, \text{Pic}^0_{Y/k}) \rightarrow 0.
\end{array}
$$

The snake lemma gives us a short exact sequence

$$
0 \rightarrow \text{NS}(X) \times \text{NS}(Y) \rightarrow \text{NS}(X \times_k Y) \rightarrow \text{Hom}_k(\text{Pic}^0_{X/k}, \text{Pic}^0_{Y/k}) \rightarrow 0.
$$

Now use that $\text{Hom}_k(A, B)$ for Abelian varieties $A, B$ over a field $k$ is a finitely generated free group, see [Mil86a, p. 122, Lemma 12.2]. □

**Lemma 5.4.** Let $X_i$, $i = 1, \ldots, n$ be connected proper varieties over an algebraically closed field $k$. If $\tilde{X}$ is an étale covering of $X_1 \times_k \ldots \times_k X_n$, there are étale coverings $\tilde{X}_i$ of $X_i$ and an étale covering $X_1 \times_k \ldots \times_k \tilde{X}_n \rightarrow \tilde{X}$.

**Proof.** By [SGA1, p. 203.f., Corollaire X.1.7], the étale fundamental group of a product of connected proper varieties over an algebraically closed field is the product of the étale fundamental groups of its factors. Now use that for an open subgroup $H \leq G$ of a profinite group $G = G_1 \times \ldots \times G_n$ contains an open subgroup $H_1 \times \ldots \times H_n$ of $G$ with $H_i \leq G_i$ open. (One can take $H_i = G_i \cap H$.) □

**Proposition 5.5.** Let $G/S$ be finite étale over $S$ connected. Then there is a connected finite étale covering $S'/S$ of degree dividing $\deg(G/S)!$ such that $G \times_S S'/S'$ is constant.

**Proof.** Choose a geometric point $s$ of $S$. Let $X$ be the $\pi_1^\text{ét}(S, s)$-set corresponding to $G/S$, and let $H \subseteq \pi_1^\text{ét}(S, s)$ be the subgroup corresponding to the elements that act as the identity on $X$, the kernel of $\pi_1^\text{ét}(S, s) \rightarrow \text{Aut}(X)$. Let $S'$ be the finite étale covering corresponding to the $\pi_1^\text{ét}(S, s)$-set $\pi_1^\text{ét}(S, s)/H$, which is connected as $\pi_1^\text{ét}(S, s)$ acts transitively on $\pi_1^\text{ét}(S, s)/H$. The scheme $G \times_S S'/S'$ is constant by [SGA1, p. 113, Corollaire V.6.5 applied to the functor $- \times_S S'$ : $\text{FÉt}/S \rightarrow \text{FÉt}/S'$ of Galois categories.

Note that $|\text{Aut}(X)| = |X| = \deg(G/S)!$, so $\deg(S'/S) = [\pi_1^\text{ét}(S, s) : H] | \deg(G/S)!$.

□

**Theorem 5.6.** Let $k$ be a field of characteristic $p$ and $S/k$ be proper, reduced and connected. Let $\mathcal{A}/S$ be a relative elliptic curve or a principally polarised Abelian scheme with constant isomorphism type of $\mathcal{A}/p$. Then there is a connected finite étale covering $S'/S$ such that $\mathcal{A} \times_S S'/S'$ is constant.

**Proof.** If $\mathcal{A}/S$ is a relative elliptic curve: Choose $N \geq 3$ such that $N$ is invertible on $S$. Since $\delta[N]/S$ is finite étale, by Proposition 5.5 there is a finite étale covering $S'/S$ such that there is an $S'$-isomorphism $\delta[N] \times_S S' \cong (\mathbb{Z}/N)^2$. Since the finite $(N \geq 3)$ moduli space $Y(N)$ of elliptic curves with full level-$N$ structure is affine by [RKM85, p. 117, Corollary 4.7.2 and 4.7.2] and $S'$ is reduced and connected, by the coherence theorem, the morphism $S' \rightarrow Y(N)$ classifying $(\delta \times_S S', \delta[N] \times_S S')$ factors over a finite extension field $k'$ of $k$. Hence $\delta[S'] \cong \delta^{\text{univ}} \times_{Y(N)} \text{Spec}(k')$ is constant.

If $\mathcal{A}/S$ is a principally polarised Abelian scheme with constant isomorphism type of $\mathcal{A}/p$: Use the same argument and use that there is a level-$n$ structure for some $n \geq 3$ not divisible by $p$ after finite étale base extension and that the Ekedahl-Oort stratification of the moduli space $\mathcal{A}_{g,1,n} \otimes_{\mathbb{F}_p} p$ for $p \nmid n$ is quasi-affine [Oort01, p. 348, Theorem 1.2]. □

**Lemma 5.7.** Let $X$ be a normal Noetherian integral scheme with function field $K = K(X)$, $\mathcal{A}$ and $\mathcal{B}$ Abelian schemes over $X$ and $L/K$ be a separable field extension. Given a homomorphism $f_L \in \text{Hom}_X(\mathcal{A}_L, \mathcal{B}_L)$, there exists a finite étale covering $X'/X$ with function field $L'$ with $L \subseteq L' \subseteq K$ and an extension of $f_L$ to $f_{X'} \in \text{Hom}_{X'}(\mathcal{A}_{X'}, \mathcal{B}_{X'})$. 
Proof. Since $X$ is normal Noetherian integral, the Abelian schemes $\mathcal{A}, \mathcal{B}$ are projective over $X$ by [Ray70], p. 161, Théorème XI.1.4. Since $X$ is Noetherian and $\mathcal{A}, \mathcal{B}$ are also flat over $X$, by [CG1’05], p. 133, Theorem 5.23, there exists the Hom-scheme $\text{Hom}_X(\mathcal{A}, \mathcal{B})$ over $X$, which is an open subscheme of the Hilbert scheme $\text{Hilb}_{\mathcal{A} \times X/\mathcal{B}/X}$, which is separated and locally of finite presentation over $X$. Since for a discrete valuation ring $R$ with quotient field $\text{Quot}(R)$, arguing as in [BLR90], p. 15, proof of Proposition 1.2/8, there is for $f_{\text{Quot}(R)} : \mathcal{A}_{\text{Quot}(R)} \to \mathcal{B}_{\text{Quot}(R)}$ a unique (by separatedness) extension to $f_R : \mathcal{A}_R \to \mathcal{B}_R$, the connected components of $\text{Hom}_X(\mathcal{A}, \mathcal{B})$ are proper over $X$. By the infinitesimal lifting criterion for unramified, $\text{Hom}_X(\mathcal{A}, \mathcal{B}) \to X$ is also unramified: Let $(R, m)$ be a local Artinian ring with residue field $k$. Then $\text{Hom}_R(\mathcal{A}_R, \mathcal{B}_R) \hookrightarrow \text{Hom}_k(\mathcal{A}_k, \mathcal{B}_k)$ is injective since $\text{Spec}(R)$ consists of a single point: Namely, if $f : \mathcal{A}_R \to \mathcal{B}_R$ maps to $f_k = 0$, $f = 0$ by the rigidity lemma [MF82], p. 115, Theorem 6.1.1). Hence any component of $\text{Hom}_X(\mathcal{A}, \mathcal{B})$ that is dominant over $X$ is finite (by Zariski’s main theorem, since it is proper and quasi-finite) and étale over $(X = A)$ integral and normal, hence geometrically unibranch, so dominant, finite and unramified implies étale by [EGAIV], p. 157, Théorème 18.10.1).

For the definition of a supersingular Abelian variety see [Oor74], p. 113, Definition 4.1. A supersingular Abelian scheme is an Abelian schemes with all fibres supersingular Abelian varieties, equivalently (for an integral base) if the generic fibre is supersingular (this follows from Theorem 5.8).

**Theorem 5.8.** Let $X$ be a normal Noetherian integral scheme of characteristic $p > 0$ and $\mathcal{A}/X$ be an Abelian scheme with supersingular generic fibre. Then there exists a finite étale covering $X' \to X$, a supersingular elliptic curve $E/F_p$ and an isogeny $(E \times F_p)_\mathfrak{p} -> \mathcal{A} \times X X'$.

**Proof.** Let $K = K(X)$ be the function field of $X$. By [Oor74], p. 113, Theorem 4.2, $\mathcal{A}_K$ is isogenous to $E^0_K$, with $E_K/K$ any (!) supersingular elliptic curve (any two supersingular elliptic curves over an algebraically closed field are isogenous, see [Oor74], p. 113). Note that for any prime $p$, there exists a supersingular elliptic curve over $F_p$, see [Mil09], p. 148f, Theorem V.4.1(c) for $p = 2$ and the text before this theorem for $p = 3$. By [Mil09], p. 146, Corollary 20.4(b) applied to the primary field extension $K/K^{\text{sep}}$, there is a separable field extension $L/K$ and an isogeny $E^0_L \to \mathcal{A}_L$. Since $E/F_p$ extends to $E \times F_p$ over $X$, the claim follows from Lemma 5.7.

**Definition 5.9.** We call an Abelian scheme $\mathcal{A}/X$ $\ell'$-isogenous if there is a proper, surjective, generically étale $\ell'$-morphism of regular schemes $f : X' \to X$ (an $\ell'$-alteration) such that $\mathcal{A} \times X X' = \mathcal{A}'$.

The following theorem about descent of finiteness of the Tate-Shafarevich group together with Theorem 4.16 implies Theorem 5.1 implies Theorem 5.4 from the introduction.

**Theorem 5.10.** Let $\ell$ be some prime invertible on $X$. Let $f : X' \to X$ be a proper, surjective, generically étale $\ell'$-morphism of regular schemes. If $\mathcal{A}$ is an Abelian scheme on $X$ such that the $\ell'$-torsion of the Tate-Shafarevich group $\text{III}(\mathcal{A}/X')$ of $\mathcal{A}' := f^* \mathcal{A} = \mathcal{A} \times X X'$ is finite, then the $\ell'$-torsion of the Tate-Shafarevich group $\text{III}(\mathcal{A}/X)$ is finite.

**Proof.** See [Kel16], p. 238, Theorem 4.29.

**Corollary 5.11.** If $X$ is a product of smooth proper curves, Abelian varieties and $K$ surfaces over a finite field of characteristic $p$ and $\mathcal{A}/X$ is (1) a relative elliptic curve or (2) an Abelian scheme such that the isomorphism type of $\mathcal{A}[p]$ is constant or (3) an Abelian scheme with supersingular generic fibre, the $\ell'$ part of the conjecture of Birch and Swinnerton-Dyer holds for $\mathcal{A}/X$ and, if $\mathcal{A}/X$ is a relative elliptic curve, $\Br(\mathcal{A})$ of non-$p$ is finite. If $X$ is a curve, the full conjecture of Birch and Swinnerton-Dyer holds for $\mathcal{A}/X$. Furthermore, the Tate conjecture holds in dimension 1 for $\mathcal{A}$.

**Proof.** The conditions (a) and (b) from Theorem 4.16 are satisfied for $S'$ in Theorem 5.6 by Example 4.20 if the base scheme is a curve or an Abelian variety as a finite étale constant connected covering of a curve or an Abelian variety is again a curve or an Abelian variety, respectively. For curves, this is clear, and for Abelian varieties see [Man77], p. 155, Theorem of Serre-Lang. So one has (a) and (b) for a product from Corollary 5.2 and Corollary 5.3. A K3 surface $X/k$ has $\pi^\ell_1(X) = \pi^\ell_1(k)$ by [Huy15], p. 11, Remark 2.3 and the homotopy exact sequence $\Gamma \to \pi^\ell_1((X \times_k k) \to \pi^\ell_1(X) \to \pi^\ell_1(k) \to 1$. Therefore, an étale covering of $X$ is of the form $X \times_k K$ with $K/k$ a finite separable field extension. Since $H^2_{Zar}(X, \mathcal{O}_X) = 0$, also $H^1(X \times_k K, \mathcal{O}_{X \times_k K}) = 0$ by [Liu06], p. 189, Corollary 5.2.27. Furthermore, $\Omega^1_{X \times_k K} = \Omega^1_{X \times_k K}$ by [Liu06], p. 271, Proposition 6.1.24(a). Now apply Theorem 5.10 to the étale covering from Lemma 5.4 to get (a) and (b) for the covering. For an Abelian scheme with supersingular generic fibre use the same argument together with Theorem 5.8 and isogeny invariance of the finiteness of the Tate-Shafarevich group [Kel16], p. 240, Theorem 4.31.
Note that $\mathcal{A}/X$ is $\ell'$-isocconstant for some $\ell' \neq \text{char}(k)$, and then we can use (a) $\implies$ (b) from Theorem \textit{2.44} to get independence from $\ell$. Using [Bau92], p. 286, Theorem 4.8, this proves the conjecture of Birch and Swinnerton-Dyer for elliptic curves with good reduction everywhere over 1-dimensional global function fields.

The finiteness of the prime-to-$p$ part of the Brauer group of the absolute variety $\mathcal{E}$ over an Abelian variety $X$ follows from the finiteness of $\text{Br}(X)[\text{non-pr}]/\text{Zar}$ and [Kel16], p. 237, Theorem 4.27. For $X$ a curve, see the proof of Corollary \textit{5.12}. For $X$ a K3 surface, see [SZ15], p. 11405, Theorem 1.3 and [Ho17], p. 1, Theorem 1.1 and note that the Brauer group of a finite field is trivial.

The Tate conjecture holds in dimension 1 since the Kummer sequence gives an exact sequence

$$0 \to \text{Pic}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^1(\mathcal{E}, \mathbb{Z}_\ell(1)) \to T_\ell \text{Br}(\mathcal{E}) \to 0$$

and $\text{Br}(\mathcal{E})[\ell^\infty]$ is finite, so $T_\ell \text{Br}(\mathcal{E}) = 0$. \hfill \square

The $p$-part will be covered in a forthcoming article [Kel17]. There, we prove that the Brauer group of an Abelian variety over a finite field is finite (including the $p$-part), descent of finiteness of the $p^\infty$-torsion of the Tate-Shafarevich group under alterations, and isogeny invariance of finiteness of the $p^\infty$-torsion of the Tate-Shafarevich group.

\textbf{Corollary 5.12.} Let $C/\mathbb{F}_q$ be a smooth proper geometrically connected curve and $\mathcal{E}/C$ be a relative elliptic curve. Then $\text{Br}(\mathcal{E}) = \text{III}(\mathcal{E}/C)$ is finite and of square order, and the Tate conjecture holds for $\mathcal{E}$.

\textbf{Proof.} This follows from [Kel16], p. 237, Theorem 4.27 and Corollary \textit{5.11} and since $\text{Br}(C) = 0$ by class field theory, see [Mil85], p. 137, Remark I.A.15 and p. 131, Theorem I.A.7 and the Albert-Brauer-Hasse-Noether theorem [NSW00], p. 437, Theorem 8.1.17.

The statement about the square order follows from [LLR05]. The Tate conjecture in dimensions other than 1 is trivial for a surface. \hfill \square

\section{6 Reduction to the case of a surface or a curve as a basis}

\textbf{Theorem 6.1.} If the analogue of the conjecture of Birch-Swinnerton-Dyer holds for all Abelian schemes over all smooth projective geometrically integral surfaces, it holds over arbitrary dimensional bases.

More precisely, if there is a sequence $S \hookrightarrow \ldots \hookrightarrow X$ of ample smooth projective geometrically integral hypersurface sections with a surface $S$ and the conjecture holds for $\mathcal{A}/S$, it holds for $\mathcal{A}/X$.

The basic idea is using ample hypersurface sections, Poincaré duality and affine Lefschetz theorem and that the conjecture of Birch and Swinnerton-Dyer depends only on $\text{III}(\mathcal{A}/X) = H^2_{et}(X, \mathcal{A})$ in dimension 1.

\textbf{Proof.} Let $Y \hookrightarrow X$ be an ample smooth geometrically connected hypersurface section (this exists by [Poo05], Proposition 2.7) with (necessarily) affine complement $U \to X$. Base changing to $k$ and writing $\tilde{X} = X \times_k k$ etc., one has by [Mil80], p. 94, Remark 3.1.30 a long exact sequence

$$\ldots \to H^i_c(U, \mathcal{A}[l^n]) \to H^i(\tilde{X}, \mathcal{A}[l^n]) \to H^i(Y, \mathcal{A}[l^n]) \to H^{i+1}_c(U, \mathcal{A}[l^n]) \to \ldots$$

(6.1)

(Note that $H^i_c(U, \mathcal{A}) = H^i(\tilde{X}, \mathcal{A})$ since $\tilde{X}/k$ is proper, and likewise for $Y$.)

Since $\mathcal{A}[l^n]/X$ is étale, Poincaré duality [Mil80], p. 276, Corollary VI.11.2 gives us

$$H^i_c(U, \mathcal{A}[l^n]) = H^{2d-i}(\tilde{U}, (\mathcal{A}[l^n])^\vee(d)).$$

(Note that the varieties live over a separably closed field.) By the affine Lefschetz theorem [Mil80], p. 253, Theorem VI.7.2, one has $H^{2d-i}(\tilde{U}, (\mathcal{A}[l^n])^\vee(d)) = 0$ for $2d - i > d$, i.e. for $i < d$. Analogously, $H^{i+1}_c(U, \mathcal{A}[l^n]) = 0$ for $i + 1 < d$. Plugging this into (6.1), one gets an isomorphism

$$H^i(\tilde{X}, \mathcal{A}[l^n]) \cong H^i(Y, \mathcal{A}[l^n])$$

(6.2)

for $i + 1 < d$. Inductively, it follows that the cohomology groups of $\tilde{X}$ in dimension $i = 0, 1$ are isomorphic to the cohomology groups of a smooth projective geometrically integral surface $(d = 2)$ $S/k$.

Since $\mathrm{cd}_k(k) = 1$, the Hochschild-Serre spectral sequence degenerates on the $E_2$-page giving exact sequences

$$0 \to H^{i-1}((\tilde{X}, \mathcal{A}[l^n])_\Gamma \to H^{i}(X, \mathcal{A}[l^n]) \to H^{i}(\tilde{X}, \mathcal{A}[l^n])_\Gamma \to 0$$

for $i + 1 < d$. Inductively, it follows that the cohomology groups of $\tilde{X}$ in dimension $i = 0, 1$ are isomorphic to the cohomology groups of a smooth projective geometrically integral surface $(d = 2)$ $S/k$.
and similar for \(S\), which implies isomorphisms \(H^i(X, \mathcal{A}[\ell^n]) \simto H^i(S, \mathcal{A}[\ell^n])\) for \(i = 0, 1\) by the 5-lemma and \([6.2]\).

It follows that there is a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & \mathcal{A}(X)/\ell^n & \to & H^1(X, \mathcal{A}[\ell^n]) & \to & \text{III}(\mathcal{A}/X)[\ell^n] & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{A}(S)/\ell^n & \to & H^1(S, \mathcal{A}[\ell^n]) & \to & \text{III}(\mathcal{A}/S)[\ell^n] & \to & 0.
\end{array}
\]

Passing to the inverse limit \(\varprojlim\), and using \(\varprojlim \mathcal{A}(X)/\ell^n = 0\) (and similar for \(S\)) because the \(\mathcal{A}(X)/\ell^n\) are finite by the weak Mordell-Weil theorem \([2.35]\) and the Néron mapping property \(\mathcal{A}(X) = A(K)\) \([2.36]\) gives a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & \mathcal{A}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \to & H^1(X, T_\ell \mathcal{A}) & \to & T_\ell \text{III}(\mathcal{A}/X) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{A}(S) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \to & H^1(S, T_\ell \mathcal{A}) & \to & T_\ell \text{III}(\mathcal{A}/S) & \to & 0.
\end{array}
\]  

(6.3)

By the snake lemma,

\[
\ker (T_\ell \text{III}(\mathcal{A}/X) \to T_\ell \text{III}(\mathcal{A}/S)) = \text{coker} (\mathcal{A}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathcal{A}(S) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)
\]

(6.4)

is a finitely generated free \(\mathbb{Z}_\ell\)-module (since \(T_\ell \text{III}(\mathcal{A}/X)\) is), so \(T_\ell \text{III}(\mathcal{A}/X) \simto T_\ell \text{III}(\mathcal{A}/S)\) iff \(\text{rk} \mathcal{A}(X) = \text{rk} \mathcal{A}(S)\).

**Proposition 6.2.** Let \(X\) be a smooth projective geometrically integral variety over a finite field of characteristic \(p\). Let \(Y \to X\) be an ample smooth projective geometrically integral hypersurface section with \(\dim Y \geq 2\) and affine complement \(U\). Let \(\mathcal{A}/X\) be an Abelian scheme. Then the restriction morphism \(\mathcal{A}(X) \to \mathcal{A}(Y)\) is an isomorphism (away from \(p\)).

**Proof.** By \([\text{Mil80}], p. 94, Remark III.1.30\), there is an exact sequence

\[
0 \to H^2_0(U, \mathcal{A}) \to H^0(X, \mathcal{A}) \to H^0(Y, \mathcal{A}) \to H^1_0(U, \mathcal{A}).
\]

The injectivity of \(\mathcal{A}(X) \to \mathcal{A}(Y)\) follows from \([6.3]\) (or \(H^2_0(U, \mathcal{A}) = 0\) since \(U\) is affine). For the surjectivity of \(\mathcal{A}(X) \to \mathcal{A}(Y)\) away from \(p\), it suffices to show that \(H^1_0(U, \mathcal{A})[\text{non}-p] = 0\) (or at least that \(H^2_0(U, \mathcal{A})[\text{non}-p]\) is finite/torsion since the cokernel is torsion-free away from \(p\) by \([6.4]\)). The Kummer exact sequence \(0 \to \mathcal{A}[\ell^n] \to \mathcal{A} \to \mathcal{A} \to 0\) with \(\ell\) invertible on \(U\) induces an exact sequence

\[
H^1_0(U, \mathcal{A}[\ell^n]) \to H^1_0(U, \mathcal{A}) \simto H^1_0(U, \mathcal{A}) \to H^2_0(U, \mathcal{A}[\ell^n]).
\]

Since \(H^1_0(U, \mathcal{A}[\ell^n]) = 0 = H^2_0(U, \mathcal{A}[\ell^n])\) because of \(\dim U > 2\) as above by Poincaré duality and the affine Lefschetz theorem, \(H^1_0(U, \mathcal{A})\) is \(\ell\)-divisible and \(\ell\)-torsion free. The exact sequence \([\text{Mil80}], p. 94, Remark III.1.30\)

\[
\mathcal{A}(X) \to \mathcal{A}(Y) \to H^1_0(U, \mathcal{A}) \to \text{III}(\mathcal{A}/X) \to \text{III}(\mathcal{A}/Y)
\]

shows, since the Mordell-Weil groups are finitely generated abelian groups by the theorem of Mordell-Weil \([2.35]\) and the Néron mapping property \(\mathcal{A}(X) = A(K)\) \([2.36]\) and the \(\ell\)-primary components of the (torsion) Tate-Shafarevich groups are cofinitely generated abelian groups by Lemma \([2.33]\) that

\[
H^1_0(U, \mathcal{A})[\text{non}-p] \cong \bigoplus_{\ell \neq p} (F_\ell \otimes (Q_\ell/\mathbb{Z}_\ell)^n) \oplus \mathbb{Z}^n
\]

with \(F_\ell\) finite abelian \(\ell\)-groups and \(n, n_\ell \in \mathbb{N}\). It follows from \(H^1_0(U, \mathcal{A})/\ell = 0\) that \(n = 0\) and then from \(H^2_0(U, \mathcal{A})[\ell] = 0\) that \(H^2_0(U, \mathcal{A})[\text{non}-p] = 0\).

It also follows from \(H^i(X, \mathcal{A}[\ell^n]) \simto H^i(S, \mathcal{A}[\ell^n])\) for \(i = 0, 1\) and Definition \([2.7]\) that \(L(\mathcal{A}/X, s) = L(\mathcal{A}/S, s)\), so if the conjecture of Birch and Swinnerton-Dyer holds for \(\mathcal{A}/S\), \(\text{rk} \mathcal{A}(X) = \text{rk} \mathcal{A}(S)\) by Proposition \([6.2]\) and \(\mathcal{A}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simto \mathcal{A}(S) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell\). Hence, the analogue of the conjecture of Birch and Swinnerton-Dyer for \(\mathcal{A}/X\) is equivalent to the conjecture for \(\mathcal{A}/S\).
Theorem 6.3. If there is a smooth projective ample geometrically integral curve $C \hookrightarrow S$ with $\text{rk } \mathcal{A}(S) = \text{rk } \mathcal{A}(C)$, the analogue of the conjecture of Birch and Swinnerton-Dyer for $\mathcal{A}/S$ is equivalent to the conjecture for $\mathcal{A}/C$.

Proof. For an ample smooth projective geometrically integral curve hypersurface section $C \hookrightarrow S$, one has still $\mathcal{A}(S)[\ell^n] \to \mathcal{A}(C)[\ell^n]$ and at least an injection $H^1(S, \mathcal{A}[\ell^n]) \hookrightarrow H^1(C, \mathcal{A}[\ell^n])$ for all $n \geq 0$ and $H^1(S, T_\ell \mathcal{A}) \hookrightarrow H^1(C, T_\ell \mathcal{A})$. Arguing in the same way as above using the commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & \mathcal{A}(S) & \otimes \mathbb{Z}_\ell & \to & H^1(S, T_\ell \mathcal{A}) & \to & T_\ell III(\mathcal{A}/S) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{A}(C) & \otimes \mathbb{Z}_\ell & \to & H^1(C, T_\ell \mathcal{A}) & \to & T_\ell III(\mathcal{A}/C) & \to & 0
\end{array}
$$

and the snake lemma

$$
\ker (T_\ell III(\mathcal{A}/S) \to T_\ell III(\mathcal{A}/C)) \to \text{coker } (\mathcal{A}(S) \otimes \mathbb{Z}_\ell \to \mathcal{A}(C) \otimes \mathbb{Z}_\ell)
$$

with $T_\ell III(\mathcal{A}/S)$ and hence the kernel being torsion-free, if the conjecture of Birch and Swinnerton-Dyer holds for $\mathcal{A}/C$ and $\text{rk } \mathcal{A}(S) = \text{rk } \mathcal{A}(C)$, the analogue of the conjecture of Birch and Swinnerton-Dyer holds for $\mathcal{A}/S$. \hfill \Box

Remark 6.4. So the question arises if there is always such a $C \hookrightarrow S \hookrightarrow \ldots \hookrightarrow X$ with $\text{rk } \mathcal{A}(S) = \text{rk } \mathcal{A}(C)$, see [GST13], Theorem 1.2 (ii) and Proposition 1.5 (iii) (over uncountable fields).

One always has the inequality $\text{rk } \mathcal{A}(S) \leq \text{rk } \mathcal{A}(C)$, so the analogue of the conjecture of Birch and Swinnerton-Dyer for $\mathcal{A}/X$ holds if there is such a $C \hookrightarrow X$ with $\text{rk } \mathcal{A}(C) = 0$, e.g. $C \cong \mathbb{P}^1_k$ and $\mathcal{A}/C$ isocostant, e.g. if $\mathcal{A}/C$ is a relative elliptic curve.

Acknowledgements. I thank the anonymous referee for significantly improving the exposition of the article, my advisor Uwe Jannsen and Maarten Derickx, Patrick Forrê, Ulrich Görtz, Walter Gubler, Peter Jossen, Moritz Kerz, Klaus Künnemann, Frans Oort, Michael Stoll, Tamás Szamuely, Georg Tamme and, from mathoverflow, abx, ACL, Angelo, anon, Martin Bright, Holger Partsch, Kestutis Cesnavicius, Torsten Ekedahl, Laurent Moret-Bailly, ndk23, Jason Starr, ulrich and xuhan; finally the Studienstiftung des deutschen Volkes for financial and idealional support.

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