Brauer groups in arithmetic geometry: exercises 9 on basic properties of the Brauer-Manin obstruction

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Let k be a global field with set of places Ω_k and X/k be a smooth projective geometrically integral variety.

Exercise 1. Let $A \in \mathrm{Br}(X)$ and $v \in \Omega_k$ be non-archimedean. Prove that

$$\begin{aligned} \mathrm{ev}_{A,v} : X(k_v) &\to \mathbf{Q}/\mathbf{Z}, \\ x_v &\mapsto \mathrm{inv}_v(x_v^* A) \end{aligned}$$

factors through $X(\mathscr{O}_v/\mathfrak{m}_v^n)$ for some $n > 0$.

Exercise 2.

1. Find an example of a variety X/k where the inclusion $X(k) \subsetneq \overline{X(k)} \subseteq X(\mathbf{A}_k)$ is strict.
2. Assume $X(\mathbf{A}_k) \neq \emptyset$ and that there is an $A \in \mathrm{Br}(X)$ and a place w such that $\mathrm{ev}_{A,w}$ is *not* constant. Prove that $\overline{X(k)} \subsetneq X(\mathbf{A}_k)$, i. e. the k-rational points are not dense in the adelic points.