Brauer groups in arithmetic geometry: exercises 8 on preparations for the Brauer-Manin obstruction

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Exercise 1. Let k be a global field with adele ring $\mathbf{A}_k = \prod'_{v \in \Omega_k} k_v = \{(x_v) \in \prod_{v \in \Omega_k} k_v : x_v \in \mathscr{O}_v \text{ for almost all } v\}$ and X/k be a smooth projective geometrically integral variety. Set $\overline{X} = X \times_k k^{\mathrm{sep}}$.

1. Show that $\mathrm{Br}_0(X) \subseteq \mathrm{Br}_1(X)$ (Hint: functoriality of the Brauer group).
2. If $X(k) \neq \emptyset$, show that the morphism $\mathrm{Br}(k) \to \mathrm{Br}(X)$ induced by the structure morphism is injective (Hint: functoriality of the Brauer group).
3. Show that $X(\mathbf{A}_k) \neq \emptyset$ is equivalent to $X(k_v) \neq \emptyset$ for all $v \in \Omega_k$ (Hint: valuative criterion of properness).
4. If $X(\mathbf{A}_k) \neq \emptyset$, show that the morphism $\mathrm{Br}(k) \to \mathrm{Br}(X)$ induced by the structure morphism is injective (Hint: Albert-Brauer-Hasse-Noether).
5. Deduce that if $X(\mathbf{A}_k) \neq \emptyset$, one has for the *arithmetic Brauer group* $\mathrm{Br}_{\mathrm{ar}}(X) := \mathrm{Br}_1(X)/\mathrm{Br}_0(X) \xrightarrow{\sim} \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$ and $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(\overline{X})^{G_k}$.
6. If $\mathrm{Pic}(\overline{X})$ is torsion-free, show that $\mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$ is finitely generated free (you may use the theorem of the base) and deduce that $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is finite (Hint: inflation-restriction exact sequence for a finite Galois trivialising extension of k).