Brauer groups in arithmetic geometry: exercises 5 on zeta functions and the Weil conjectures

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Exercise 1 (Weil conjectures for projective space)**.** Verify the Weil conjectures for $\mathbf{P}^n_{\mathbf{F}_q}$ (this exercise works more generally for varieties with an stratification by affine spaces like Graßmannians):

Recall that for $X/\mathbf{F}_q$ of finite type (so $X(\mathbf{F}_{q^i})$ is a finite set for all $i > 0$),

$$Z(X/\mathbf{F}_q, T) := \exp\left(\sum_{n\geq 1} |X(\mathbf{F}_{q^n})| \frac{T^n}{n}\right) \in \mathbf{Q}[\![T]\!]$$

and its zeta function is $\zeta(X/\mathbf{F}_q, s) := Z(X/\mathbf{F}_q, q^{-s})$ for $s \in \mathbf{C}$ where it converges.

(*Remark:* One can also define the arithmetic zeta function of $X/\mathbf{Z}$ of finite type

$$\zeta(X, s) := \prod_{x\in|X|} \frac{1}{1 - |\kappa(x)|^{-s}}$$

for $s \in \mathbf{C}$ with $\mathrm{Re}(s) \gg 0$ and prove that this gives the same function for varieties over finite fields. Note that $\zeta(\operatorname{Spec}\mathbf{Z}, s)$ is the Riemann zeta function and, more generally, $\zeta(\mathcal{O}_K, s)$ is the Dedekind zeta function of a number field K. For the following exercises, use the first definition.)

1. Prove that $\zeta(X/\mathbf{F}_q, s)$ only depends on the scheme X, so one can write $\zeta(X, s)$.
2. Let $X/\mathbf{F}_q$ be of finite type and $X = X_1 \coprod X_2$ (as sets) with X_1, X_2 locally closed subschemes of X. Prove that $\zeta(X, s) = \zeta(X_1, s) \cdot \zeta(X_2, s)$.
3. Find
$$Z(\mathbf{A}^n_{\mathbf{F}_q}/\mathbf{F}_q, T) \in \mathbf{Q}[\![T]\!] \cap \mathbf{Q}(T)$$
and prove that $\zeta(\mathbf{A}^n_{\mathbf{F}_q}, s)$ is a rational function in q^{-s}.
4. (Rationality) Find $\zeta(\mathbf{P}^n_{\mathbf{F}_q}, s)$ using part 2 and 3. Where is it holomorphic?
5. (Functional equation) Find the functional equation for $\zeta(\mathbf{P}^n_{\mathbf{F}_q}, s)$ relating $\zeta(\mathbf{P}^n_{\mathbf{F}_q}, s)$ and $\zeta(\mathbf{P}^n_{\mathbf{F}_q}, \dim \mathbf{P}^n_{\mathbf{F}_q} - s)$.
6. (Riemann hypothesis) Find the zeroes and poles of $\zeta(\mathbf{P}^n_{\mathbf{F}_q}, s)$.
7. (Betti numbers) Compare with the singular (Betti) cohomology of the analytic space associated to $\mathbf{P}^n_{\mathbf{C}} = \mathbf{P}^n_{\mathbf{Z}} \times_{\mathbf{Z}} \mathbf{C}$. Note that $\mathbf{P}^n_{\mathbf{Z}}/\mathbf{Z}$ is proper and smooth, so proper-smooth base change for étale cohomology applies.