Brauer groups in arithmetic geometry: exercises 4 on cohomology of schemes

Timo Keller

November 5, 2018

Exercise 1 (Leray spectral sequence)**.**

1. Show that injective sheaves are flasque using $0 \to j_! \mathbf{Z}_U \to \mathbf{Z}_X$ for $j : U \hookrightarrow X$ an open immersion.
2. Let $f : X \to Y$ be a morphism of topological spaces. Show that f_* maps flasque sheaves to flasque sheaves, which are acyclic.
3. Let $f : X \to Y$ be a morphism of topological spaces and $\mathscr{F}$ an abelian sheaf on X. Show that there is a spectral sequence $\mathrm{H}^p(Y, \mathrm{R}^q f_* \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$, called the Leray spectral sequence. (This can be generalised to morphisms of sites.)
4. Let $i : Z \hookrightarrow X$ be a closed immersion and $\mathscr{F}$ an abelian sheaf on Z. Show that the Leray spectral sequence degenerates to $\mathrm{H}^p(X, i_* \mathscr{F}) = \mathrm{H}^p(Z, \mathscr{F})$. (We will see that finite morphisms have trivial higher direct images for the étale topology.)

Exercise 2 (divisor exact sequence)**.** Let X be a Noetherian integral regular scheme. Show that there is an exact sequence

$$1 \to \mathscr{O}_X^\times \to \mathscr{M}_X^\times \overset{\mathrm{div}}{\to} \mathscr{Z}_X^1 \to 0$$

of sheaves on X.

Exercise 3 (Zariski cohomology of $\mathbf{G}_m$)**.** Let X be a Noetherian integral regular scheme. Using the previous exercises and that cohomology on Noetherian topological spaces commutes with filtered colimits, show that

$$\mathrm{H}^i(X, \mathcal{O}_X^\times) = \begin{cases} \Gamma(X, \mathscr{O}_X)^\times, & i = 0 \\ \mathrm{Pic}(X), & i = 1 \\ 1, & i > 1. \end{cases}$$

(*Remark*: $\mathrm{H}^i(X_{\text{ét}}, \mathbf{G}_m) \neq 1$ for $i > 1$ in general, even for fields: $\mathrm{H}^2(k_{\text{ét}}, \mathbf{G}_m) = \mathrm{Br}(k)$)