Brauer groups in arithmetic geometry: exercises 3 on Brauer groups of fields

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On this exercise sheet, you see the cohomological machinery in action.

Exercise 1 (n-torsion of Brauer groups)**.** Let k be a field with n be invertible in k. Show that $\mathrm{Br}(k)[n] = \mathrm{H}^2(k, \mu_n)$. (Hint: Kummer sequence and Hilbert 90.)

Exercise 2 (exact sequence of Brauer groups)**.** Let L/k be a Galois extension of k. Show that

$$0 \to \mathrm{Br}(L/k) \to \mathrm{Br}(k) \to \mathrm{Br}(L)^{\mathrm{Gal}(L/k)}$$

is exact and that $\mathrm{Br}(k) = \varinjlim_{L/k \text{ finite Galois}} \mathrm{Br}(L/k)$. (Hint: Beginning of the 5-term exact sequence and Hilbert 90.)

Exercise 3 (Brauer group of cyclic extensions)**.** Let L/k be a finite cyclic extension. Show that $\mathrm{Br}(L/k) = k^\times / \mathrm{N}_{L/k} L^\times$. (Hint: Recall the calculation of the Brauer group of a real closed field and the 2-periodicity of Group cohomology of finite cyclic groups.)

Exercise 4 (Brauer group of the maximal abelian extension of a global field)**.** Let k be a global field. Show that $\mathrm{Br}(k^{\mathrm{ab}}) = 0$. (Hint: Use without proof that for every $x \in \mathrm{Br}(k)$, there is a finite cyclotomic extension L/k with $\mathrm{res}_{L/k}(x) = 0 \in \mathrm{Br}(L)$.)

Exercise 5 (skew fields over $\mathbf{R}$)**.** Show that every non-trivial skew field over $\mathbf{R}$ (or, more generally, a real closed field) is isomorphic to the Hamilton quaternions $\mathbf{H} = (-1,-1)_{-1}$. (Hint: Look at the proof of Wedderburn's theorem using $\mathrm{Br}(\mathbf{F}_q) = 0$.)

Exercise 6 (Milnor K-theory of $\mathbf{R}$)**.** Calculate $\mathrm{K}_n^{\mathrm{M}}(\mathbf{R})$ for all $n \geq 0$. (Hint: Modify the proof of the calculation of the Milnor K-theory of an algebraically closed field and use $\mathbf{R}^\times = \{\pm 1\} \times \mathbf{R}$ with $\mathbf{R}$ uniquely divisible.)