Brauer groups in arithmetic geometry: exercises 2 on group and Galois cohomology

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Exercise 1 (examples of group cohomology)**.** Find an example of an infinite profinite group G and a finite discrete G-module $A \neq 0$ such that

1. $\mathrm{H}^1(G, A) = 0$,
2. $\mathrm{H}^1(G, A) \neq 0$,
3. $\mathrm{H}^2(G, A) = 0$,
4. $\mathrm{H}^2(G, A) \neq 0$.

Is there a short exact sequence $0 \to A' \to A \to A'' \to 0$ such that $A^G \to A''^G$ is surjective and $\mathrm{H}^1(G, A') \neq 0$?

Exercise 2 (vanishing of group cohomology)**.** Let G be a finite group and A a G-module. The proof of Lemma 1.1.10 shows that $|G|\mathrm{H}^i(G, A) = 0$ for $i > 0$. If $nA = 0$ (A is of **exponent** n; this is in particular true for $n = |A|$ if A is finite), show that $n\mathrm{H}^i(G, A) = 0$ for all $i \geq 0$. Conclude that if $(|G|, n) = 1$, then $\mathrm{H}^i(G, A) = 0$ for $i > 0$.

Note: It follows from this applied to $\mathrm{H}^2(G, A) = \mathrm{EXT}(G, A)$ that every extension $0 \to A \to E \to G \to 1$ is **split**, i. e. $E \cong A \rtimes G$ if $(|G|, |A|) = 1$. One can show from this that this holds also for non-abelian A with $(|G|, |A|) = 1$.

Exercise 3 (five term exact sequence)**.** Let $E_2^{p,q} \Rightarrow E^{p+q}$ be a first quadrant E_2-spectral sequence.

1. Show that
$$0 \to E_2^{1,0} \xrightarrow{\kappa_1} E^1 \xrightarrow{\kappa_2} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{\kappa_1} E^2$$
is exact.
2. More generally, if $E_2^{p,q} = 0$ for all $0 < q < n$ and all p, show that $\kappa_1 : E_2^{p,0} \xrightarrow{\sim} E^p$ for all $p < n$ and
$$0 \to E_2^{n,0} \xrightarrow{\kappa_1} E^n \xrightarrow{\kappa_2} E_2^{0,n} \xrightarrow{d} E_2^{n+1,0} \xrightarrow{\kappa_1} E^{n+1}$$
is exact.
3. What happens if the spectral sequence has only two non-zero columns, i. e. $E_2^{p,q} = 0$ for $p \neq 0, n$?

Exercise 4 (Hilbert 90 for SL_n)**.** Show that $\mathrm{H}^1(L/k, \mathrm{SL}_n(L)) = \{*\}$ for all Galois extensions L/k. (Hint: Find a short exact sequence containing $\mathrm{SL}_n(L)$ and use Hilbert 90 for $\mathrm{GL}_n(L)$.)