Brauer groups in arithmetic geometry: exercises 10 on the Brauer-Manin set of a del Pezzo surface of degree 4

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January 17, 2019

Exercise 1 (Swinnerton-Dyer)**.** Let $X \hookrightarrow \mathbf{P}^4_{\mathbf{Q}}$ be defined by the equations $x^2-5y^2 = uv$, $x^2-5z^2 = (u+v)(u+2v)$.

1. Prove that $X/\mathbf{Q}$ is a smooth projective geometrically integral surface.

2. Determine the canonical bundle $\omega_{X/\mathbf{Q}}$ to show that X is a del Pezzo surface of degree 4 in $\mathbf{P}^4_{\mathbf{Q}}$, in particular geometrically rational.

3. Find a finite set S of primes outside of which X has good reduction.

4. Prove that $X(\mathbf{A}_{\mathbf{Q}}) \neq \emptyset$: For $v \neq 2$, one of $5, -5, -1$ is a square in $\mathbf{Q}_v$ by finding a $\mathbf{Q}_v$-rational point with $u = v = 0$, $u = 1, v = 0$ and $u = v = 1$, respectively; for $p = 2$ use $u = -5, v = 1$. Alternatively, use the Weil conjectures and a CAS for finitely many places.

5. Prove that $A := (5, \frac{u+v}{u}) = (5, \frac{u+v}{v}) = (5, \frac{u+2v}{u}) = (5, \frac{u+2v}{v})$ is a 2-torsion element of the Brauer group of the function field K of X by showing that the pairwise products of these elements are trivial using identities like $\frac{u+v}{u} \cdot \frac{u+v}{v} = \mathrm{N}_{K(\sqrt{5})/K}\left(\frac{u+v}{x+\sqrt{5}y}\right)$.

6. Prove that $A \in \mathrm{Br}(X)[2]$ since the common locus of the zeros and poles of the occurring rational functions has codimension 2 in X.

7. Prove that $\mathrm{ev}_{A,v}$ vanishes on $X(\mathbf{Q}_v)$ for $v \neq 5$: Distinguish the cases 5 a square in $\mathbf{Q}_v$ or not, use a suitable representative of A and that $\mathrm{ev}_{A,v}$ is locally constant. Alternatively, use a smooth projective geometrically integral model of X over $\operatorname{Spec} \mathbf{Z} \setminus S$, whose Brauer group A extends to.

8. Prove that $\mathrm{ev}_{A,5}$ takes on the constant value $\frac{1}{2}$ on $X(\mathbf{Q}_5)$.

9. Conclude that $X(\mathbf{Q}) = \emptyset$ and thus X is not rational.