Brauer groups in arithmetic geometry: exercises 1 on profinite groups and discrete G-modules

Timo Keller

October 15, 2018

Exercise 1 (equivalent conditions for profinite groups)**.** Show that for a topological group G the following are equivalent:

1. G is an inverse limit of finite discrete groups.

2. G is a Hausdorff compact totally disconnected group.

Hint for $2 \implies 1$: Show that the natural homomorphism $G \to \varprojlim_N G/N$ is an isomorphism of topological groups where N runs through the open normal subgroups of G. If U is an open and closed neighbourhood of 1 in G, let $U' = \{u \in U : Uu \subseteq U\}$ and $U'' = \{u \in U' : u^{-1} \in U'\}$ and show that U'' is an open subgroup.

Exercise 2 (equivalent conditions for discrete G-modules)**.** Show that for a profinite group G and an abelian continuous G-module A the following are equivalent:

1. $G \times A \to A, (g, a) \mapsto ga$ is continuous with the discrete topology on A.

2. $A = \bigcup_U A^U$ where U runs through the open subgroups of G.

3. For every $a \in A$, the stabiliser $G_a \subseteq G$ is open.

Exercise 3 (examples of discrete Galois modules)**.** Let L/k be (not necessarily finite) Galois and $G = \mathrm{Gal}(L/k)$. Show that L and $L^\times$ are discrete G-modules.

Exercise 4 (infinite Galois theory)**.** Give an example of two different subgroups of $\mathrm{Gal}_{\mathbf{F}_q}$ (one of them necessarily non-closed) which have the same fixed field.

Exercise 5 (complex representations of profinite groups)**.** Let G be a profinite group.

1. Let $H \subseteq G$ be an open subgroup. Then $[G : H] < \infty$.

2. Let $\varphi : G \to S^1$ be a continuous homomorphism. Then $\mathrm{im}(\varphi)$ is finite. (Hint: There is an open neighbourhood of identity in S^1 containing no non-trivial subgroup.)

3. Let $\varphi : G \to \mathrm{GL}_n(\mathbf{C})$ be a continuous homomorphism. Then $\mathrm{im}(\varphi)$ is finite. (Hint: Use that every closed subgroup of a Lie group is again a Lie group. Alternative: There is an open neighbourhood of identity in $\mathrm{GL}_n(\mathbf{C})$ containing no non-trivial subgroup.)

4. Can you find a topological field with a non-trivial topology other than $\mathbf{C}$ such that there is a representation with infinite image?